

Unit 10

Integrating scalar and vector fields

Introduction

Unit 8 explained how to carry out area integrals over flat surfaces, surface integrals over curved surfaces, and volume integrals in three-dimensional space. However, there is another type of integral that is important to us – an integral along a curve, known as a *line integral*.

You are familiar with the idea of integrating along a straight line. Figure 1 shows a straight rod lying along the x -axis with one end at $x = 0$ and the other end at $x = L$. The rod may have an uneven distribution of mass. Its linear density (its mass per unit length) is then a function of position, which we denote by $\lambda(x)$. The mass of a short segment of the rod between x and $x + \delta x$ is given by

$$\delta m \simeq \lambda(x) \delta x,$$

and the mass of the whole rod is found by adding up the masses of all of its segments. In the limit where the rod is divided into an infinite number of infinitesimally short segments, the sum becomes an integral and the total mass of the rod is expressed as

$$M = \int_0^L \lambda(x) dx.$$

A new feature introduced in this unit is to allow the rod to be curved (Figure 2). The task of finding the mass of a curved rod can again be approached by dividing the rod into many short segments, finding the mass of each segment, and adding all the contributions together. In the limit where the segments become infinitesimally short, the sum becomes an integral. However, this integral is not along the x -axis, but is along the curved path occupied by the rod. Such an integral is called a *line integral*.

There are a couple of problems that must be solved before we can evaluate a line integral of this sort. When we split the curved rod into segments, we must have a way of labelling the different segments. This is done by choosing a parameter t that increases smoothly from one end of the rod to the other. We can then talk about a segment of the rod for which the parameter has values between t and $t + \delta t$. We also need to find an expression for the length of a segment in terms of t and δt . Once these problems have been solved, the line integral can be expressed as a definite integral over the parameter t . You will see how this works in Section 1.

In Section 2, you will also see how to define and evaluate the line integrals of vector fields. The idea is simple enough: at each point along the path, we take the component of the field that is parallel to the path, and then integrate this component along the path. This turns out to be a very powerful tool in physical applications.

For example, when you lift an object, you expend energy working against gravity (Figure 3) and the lifted object gains energy as a result. The object may be lifted straight upwards, or in a circular arc, or in a spiral – however you like. Whichever path is chosen, the energy transferred to the object by

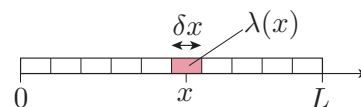


Figure 1 A straight rod

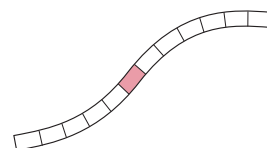


Figure 2 A curved rod



Figure 3 Lifting weights

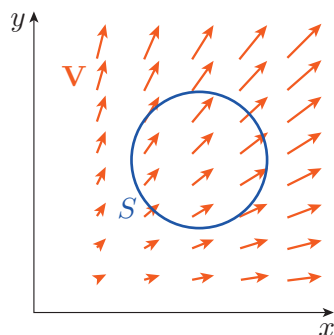


Figure 4 A diverging vector field \mathbf{V} and a spherical surface S (cross-sectional view)

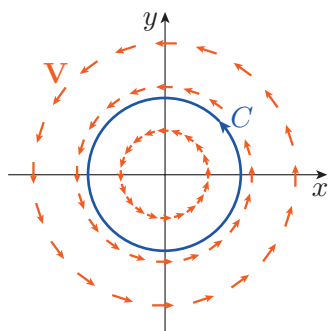


Figure 5 A circulating vector field \mathbf{V} and a closed path C

the applied force is given by the line integral of the force along the object's path. This is the line integral of a vector field. Such calculations give physicists a precise language for talking about energy transfers.

You might ask whether the energy needed to move an object from one point A to another point B depends on the detailed path taken, or just the points A and B . The answer depends on the forces acting. In mathematical terms, we need to find the conditions under which line integrals depend only on their start and end points. Section 3 will show that some types of vector field have line integrals that are always path-independent. Such fields are said to be *conservative*.

The remainder of the unit uses integrals to get a deeper, and more powerful, understanding of divergence and curl. You will remember that Unit 9 made tentative interpretations of divergence and curl: divergence was associated with the local outflow of a vector field, and curl with a local rotation or swirling. In this unit we will be more precise.

Figure 4 shows a vector field \mathbf{V} , together with an imaginary spherical surface S . Let us suppose that the field represents the flow of a fluid. Then it is apparent from the diagram that there is a net flow outwards, from the inside of the sphere into the exterior space. We can quantify this outward flow by integrating the outward normal component of the field over the surface of the sphere. This involves a type of surface integral called the *flux* of the vector field. We will use this idea to establish a precise link between divergence and outflow.

Something similar can be achieved for curl. Figure 5 shows a vector field \mathbf{V} that represents a different type of fluid flow – one that circulates rather than radiates outwards. The figure also shows an imaginary closed path C , traversed in an anticlockwise sense (as indicated by the blue arrow). Then we can calculate the line integral of the vector field \mathbf{V} around the path C . This gives a quantity called the *circulation* of the field around C . We will use this idea to establish a precise link between curl and rotation.

The links between flux and divergence, and between circulation and curl, are encapsulated by two important theorems – the *divergence theorem* and the *curl theorem* – which are discussed and used in Sections 4 and 5. These theorems give physicists and engineers powerful tools for exploring real phenomena involving fields, and are essential knowledge for anyone who wants to understand electromagnetic fields or fluid flow.

Study guide

This unit builds on all the previous units in this book. You need to be familiar with partial derivatives from Unit 7, surface and volume integrals from Unit 8, and gradients, divergences and curls from Unit 9.

The first half of the unit deals with line integrals of various types. Section 1 covers line integrals of scalar fields, while Section 2 covers line integrals of vector fields. Special vector fields with path-independent line integrals are discussed in Section 3.

The second half of the unit uses integrals to gain more powerful insights into divergence and curl. Section 4 defines the flux of a vector field, and relates it to divergence using the divergence theorem. Section 5 defines the concept of circulation, and relates it to curl using the curl theorem.

1 Line integrals of scalar fields

Imagine a whale taking a meandering journey through the ocean. Within the ocean there are huge numbers of plankton, the whale's staple food supply. The plankton are distributed unevenly, so the number of plankton per unit volume is a function $n(\mathbf{r})$ of position \mathbf{r} .

Let us break down the whale's journey into many short steps or segments. The i th step starts at position \mathbf{r}_i and is of length δl_i . The number of plankton encountered by the whale during this step is

$$\delta N_i \simeq n(\mathbf{r}_i) A \delta l_i,$$

where A is the area of the whale's open mouth (assumed to be permanently open). We can also write this as

$$\delta N_i \simeq \lambda_i \delta l_i,$$

where $\lambda_i = n(\mathbf{r}_i) A$ is the number of plankton per unit length that are within range of the whale's open mouth in step i .

The total number of plankton N encountered by the whale during its journey is found by adding together contributions from each step, so

$$N \simeq \sum_i \lambda_i \delta l_i.$$

This is an approximation because the number of plankton per unit length may vary *within* a step. We really ought to consider the sum in the limit of an infinite number of steps, each of vanishingly small size. In this limit, the sum becomes an integral and the total number of plankton encountered is written as

$$N = \int_C \lambda(\mathbf{r}) dl.$$

An integral like this is called a **line integral**. Of course, we have not told you how to evaluate such an integral – we will do that shortly; for the moment, we focus on the concept and the notation that expresses it.

Notice that the integral sign does not have lower and upper limits.

Instead, it carries the symbol C , which labels the whale's path. In general, the detailed path C matters, not just its start and end points. Along some paths between given start and end points, a whale may encounter many plankton; along others, it may find very few.

Another example of a line integral arises when we calculate the length of a path. How far does the whale swim as it travels along a path C ? We again divide the path into many segments. The total length of the path is then approximated by

$$L \simeq \sum_i \delta l_i,$$

where δl_i is the length of the i th segment.

Taking the limit of an infinite number of segments, each of vanishingly small length, the total length of the path is given by the line integral

$$L = \int_C 1 \, dl.$$

The notation is deceptively simple. Remember that the integral is along the path C , and of course the length obtained depends on the path. In general, you cannot just do the integral over l and then substitute in limits. There is a special technique for doing line integrals, which we now explain.

1.1 Line integrals in Cartesian coordinates

In this subsection, we discuss line integrals of scalar fields along given *paths*, calculated using Cartesian coordinates (x, y, z) .

A path is more than a curve!

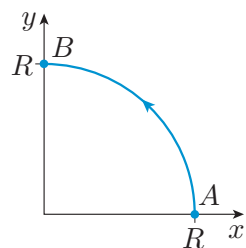


Figure 6 An anticlockwise path around a quarter-circle

A **path** is a curve with a definite sense of progression from a start point to an end point. In diagrams, the sense of progression may be indicated by an arrow, as in Figure 6.

Parametric representation of a path

The concept of a line integral along a path is based on the idea of splitting the path into many short segments. In the discussion above we labelled the segments by an index i , but this is not convenient for calculations. Instead, we choose a continuous parameter t that increases smoothly from the beginning of the path to the end. We can then talk about a segment of the path for which the parameter has values between t and $t + \delta t$.

We must specify the shape of the path. This is done by expressing the coordinates of points on the path as functions of the parameter t . For a path in three-dimensional space, we write

$$x = x(t), \quad y = y(t), \quad z = z(t) \quad (t_1 \leq t \leq t_2),$$

where $t = t_1$ at the start point and $t = t_2$ at the end point. Such a set of equations is said to give a **parametric representation**, or a **parametrisation**, of the path. For two-dimensional paths in the xy -plane, $z(t) = 0$, but this is usually left out of the description.

The minimum and maximum values of t always correspond to the start and end points of the path, respectively.

Let us take the path in Figure 6 as an example. This path starts at the point $A = (R, 0)$ on the x -axis, progresses anticlockwise around a circular arc of radius R , and ends at the point $B = (0, R)$ on the y -axis. You can imagine an insect crawling along this path, starting from A at time $t = 0$, and travelling at a steady rate until it reaches B at time $t = \pi/2$. Then at time t , the x - and y -coordinates of the insect are

$$x(t) = R \cos t, \quad y(t) = R \sin t \quad (0 \leq t \leq \pi/2). \quad (1)$$

These equations provide a parametric representation of the path. They can also be written in the vector form

$$\mathbf{r}(t) = R \cos t \mathbf{i} + R \sin t \mathbf{j},$$

where $\mathbf{r}(t)$ is the position vector of the insect at time t . The relationship between $\mathbf{r}(t)$ and the components in equations (1) is shown in Figure 7.

The parameter t always increases along the path, so in our analogy, the insect always moves forwards. However, the insect could progress in a variety of ways, giving a variety of parametric representations. For example, we could have

$$x(t) = R \cos(t^2), \quad y(t) = R \sin(t^2) \quad (0 \leq t \leq \sqrt{\pi/2}). \quad (2)$$

This is an equally valid parametric representation of the path in Figure 6, but corresponds to a non-uniform rate of progression.

The picture of an insect crawling along a path is just a device to aid understanding. The parameter t need not represent time – it could be any quantity that increases along the path. In particular, if we consider the path traced out by a whale as it swims through the ocean, the parametric representation of this path need not describe the location of the whale as a function of time!

Each parametrisation defines a certain path. The following parametric equations correspond to the path in Figure 8, which is a quarter-circle traversed *clockwise*:

$$x(t) = R \sin t, \quad y(t) = R \cos t \quad (0 \leq t \leq \pi/2).$$

This is not the same as the path in Figure 6 because the sense of progression has been reversed.

In some cases, the start point and end point are identical. For example, the equations

$$x(t) = R \cos t, \quad y(t) = R \sin t \quad (0 \leq t \leq 2\pi)$$

represent the circular path shown in Figure 9, which starts and ends at the point $(R, 0)$ on the x -axis. In general, paths that have distinct start and end points are said to be **open**, while paths with identical start and end points are said to be **closed**. Any journey where you leave your house in the morning and return in the evening is a closed path.

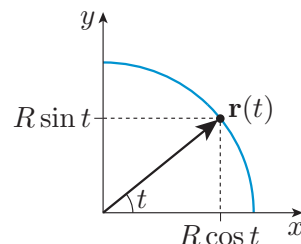


Figure 7 Cartesian components of a point on a path

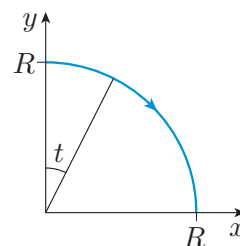


Figure 8 A clockwise path around a quarter-circle

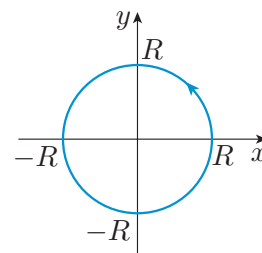


Figure 9 An anticlockwise path around a circle

The length of a short segment of a path

Figure 10 shows a short segment of a path that lies in the xy -plane. This segment begins at point P , with parameter value t and coordinates (x, y) , and ends at point Q , with parameter value $t + \delta t$ and coordinates $(x + \delta x, y + \delta y)$.

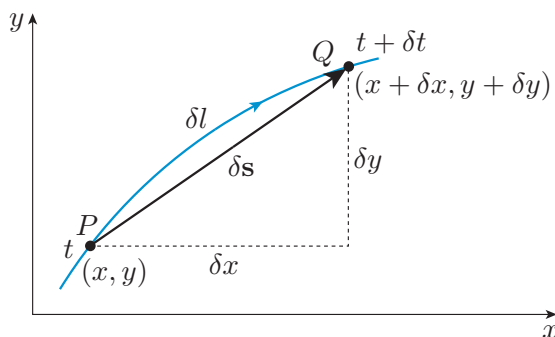


Figure 10 A short segment of a path

The displacement vector from P to Q is given by

$$\delta \mathbf{s} = \delta x \mathbf{i} + \delta y \mathbf{j}. \quad (3)$$

Dividing and multiplying the right-hand side by δt , we get

$$\delta \mathbf{s} = \left(\frac{\delta x}{\delta t} \mathbf{i} + \frac{\delta y}{\delta t} \mathbf{j} \right) \delta t \simeq \left(\frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} \right) \delta t, \quad (4)$$

where the last step follows because we are assuming that δt is very small. The magnitude of this displacement is the square root of the sum of the squares of the components. Because $\delta t > 0$, we get

$$|\delta \mathbf{s}| \simeq \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} \delta t.$$

This approximates the *distance* between the points P and Q . We are interested in the *curved segment* of path between P and Q , which has length δl . However, if the curve is reasonably smooth, and P and Q are very close together, then δl is well approximated by $|\delta \mathbf{s}|$. In the limit where the points P and Q approach one another, any error introduced by this approximation becomes negligible. We can therefore say that the length of a tiny segment of the path, with parameter values between t and $t + \delta t$, is

$$\delta l \simeq \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} \delta t. \quad (5)$$

This result applies to any path confined to the xy -plane.

For a path in three-dimensional space, we go through a similar argument but include the z -coordinate. The displacement vector then becomes

$$\delta \mathbf{s} = \delta x \mathbf{i} + \delta y \mathbf{j} + \delta z \mathbf{k},$$

Recall that if $\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j}$,
then $|\mathbf{a}| = \sqrt{a_x^2 + a_y^2}$.

and the expression for the length of a tiny segment of the path is

$$\delta l \simeq \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \delta t. \quad (6)$$

The length of a path

The simplest use of line integrals is to find the total length of a path. We begin with the two-dimensional case.

Suppose that we have a path C in the xy -plane, starting at point A and ending at point B . To find the length of this path, we must add up the lengths of all of its segments, taking the limit of an infinite number of segments, each of infinitesimal length. In this limit, the sum becomes an integral, and the total length L of the path is given by the following definite integral, where t_1 and t_2 are the parameter values at the start and end of the path.

Length of a path in the xy -plane

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt. \quad (7)$$

Example 1

Use the parametrisation of equations (1) to find the length of the quarter-circle path in Figure 6.

Solution

The parametric equations are

$$x(t) = R \cos t, \quad y(t) = R \sin t \quad (0 \leq t \leq \pi/2),$$

so

$$\frac{dx}{dt} = -R \sin t \quad \text{and} \quad \frac{dy}{dt} = R \cos t.$$

Hence

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (-R \sin t)^2 + (R \cos t)^2 \\ &= R^2(\sin^2 t + \cos^2 t) = R^2. \end{aligned}$$

Using equation (7), the total length of the path is

$$L = \int_0^{\pi/2} \sqrt{R^2} dt = \int_0^{\pi/2} R dt = \frac{\pi}{2} R,$$

as expected for a quarter of the circumference of a circle of radius R .

We can calculate the length of the same path using the alternative parametrisation of equations (2). In this case, the parametric equations are

$$x(t) = R \cos(t^2), \quad y(t) = R \sin(t^2) \quad (0 \leq t \leq \sqrt{\pi/2}),$$

and the chain rule of ordinary differentiation gives

$$\frac{dx}{dt} = -R \sin(t^2) \times 2t \quad \text{and} \quad \frac{dy}{dt} = R \cos(t^2) \times 2t.$$

Hence

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4R^2 t^2 (\sin^2(t^2) + \cos^2(t^2)) = 4R^2 t^2,$$

and the total length of the path is

$$L = \int_0^{\sqrt{\pi/2}} 2Rt \, dt = 2R \left[\frac{1}{2}t^2\right]_0^{\sqrt{\pi/2}} = \frac{1}{2}\pi R,$$

as before.

Any valid parametrisation can be used, and the answer will always be the same, but some choices make life easier than others!

A line integral along a given path does not depend on the choice of parametrisation. You are free to choose any parametrisation you like, provided that it gives a correct representation of the path.

You need not worry about which parametrisation to use; where it is not obvious, we will always suggest an appropriate choice.

We can also consider the reverse path, shown in Figure 8. This occupies the same quarter-circle curve as before, but the start and end points are interchanged, so the path is traversed in the reverse sense. This reverse path is parametrised by the equations

$$x(t) = R \sin t, \quad y(t) = R \cos t \quad (0 \leq t \leq \pi/2).$$

It is intuitively obvious that this path must have the same length as before, and this can be easily verified. We have

$$\frac{dx}{dt} = R \cos t, \quad \frac{dy}{dt} = -R \sin t,$$

so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (R \cos t)^2 + (-R \sin t)^2 = R^2,$$

giving

$$L = \int_0^{\pi/2} R \, dt = \frac{1}{2}\pi R,$$

as before.

In general, the lengths of curves, and the line integrals of scalar functions, do not depend on the sense in which a path is traversed.

Why do we bother to distinguish between curves and paths? You will see that this distinction *does* matter for the line integrals of vector fields. Our terminology bears this in mind.

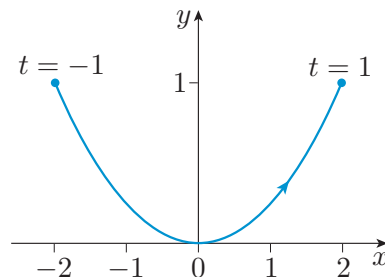
Exercise 1

The parabolic arc shown in the margin has parametric representation

$$x(t) = 2t, \quad y(t) = t^2 \quad (-1 \leq t \leq 1).$$

What is the length of this arc? You may use the standard integral

$$\int \sqrt{1+x^2} dx = \frac{1}{2} \left(x\sqrt{1+x^2} + \ln(x + \sqrt{1+x^2}) \right) + C.$$



The method is easily extended to paths in three-dimensional space. In this case, the parametric representation gives x , y and z as functions of the parameter t . The formula for the length of a segment is given by equation (6), and the total length of the path is given by the following expression.

Length of a path in three dimensions

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt, \quad (8)$$

where t_1 and t_2 are the parameter values at the start and end of the path.

Exercise 2

One turn of a helical path has the parametric representation

$$x = a \cos t, \quad y = a \sin t, \quad z = bt \quad (0 \leq t \leq 2\pi),$$

where a and b are positive constants. Find the length of this path.

Line integrals of scalar functions

We sometimes need to integrate a scalar function along a given path. For example, we might find the total mass of a curved rod by integrating its linear density (its mass per unit length) along the rod.

Starting at one end of the rod, we follow a path C that tracks along the rod and stops at the other end. Then the total mass of the rod can be expressed as the line integral

$$M = \int_C \lambda dl,$$

where λ is the linear density at points along the curved rod.

In two dimensions, the path is described by parametric equations

$$x = x(t), \quad y = y(t) \quad (t_1 \leq t \leq t_2),$$

and the linear density of the rod is given by a function

$$\lambda = \lambda(x(t), y(t)).$$

We also know that a short segment of the rod has length

$$\delta l \simeq \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \delta t.$$

This leads to the following expression for the total mass of the rod.

$$M = \int_{t_1}^{t_2} \lambda(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt. \quad (9)$$

A similar formula applies in three dimensions, with obvious adjustments to include $z(t)$ and $(dz/dt)^2$. Formulas like this also apply to other quantities that are given per unit length of a path. For example, λ could represent the number of accessible plankton per unit length along a whale's path.

Example 2

A non-uniform curved rod lies in the xy -plane. Its coordinates (in metres) are given by the parametric equations

$$x(t) = 2t, \quad y(t) = 1 - t^2 \quad (-1 \leq t \leq 1).$$

The linear density of the rod (in kilograms per metre) is

$$\lambda(x, y) = \frac{1}{(x^2 + y^2)^{1/2}}.$$

What is its mass?

Solution

From the given parametric equations, we have

$$x^2(t) + y^2(t) = 4t^2 + (1 - t^2)^2 = t^4 + 2t^2 + 1 = (1 + t^2)^2,$$

so

$$\lambda(x(t), y(t)) = \frac{1}{1 + t^2}.$$

Also,

$$\frac{dx}{dt} = 2 \quad \text{and} \quad \frac{dy}{dt} = -2t,$$

so

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 2\sqrt{1 + t^2}.$$

The mass of the rod is therefore

$$M = \int_{-1}^1 \frac{1}{1+t^2} \times 2\sqrt{1+t^2} dt = 2 \int_{-1}^1 \frac{1}{\sqrt{1+t^2}} dt.$$

Using a standard integral given in the Handbook, we get

$$\begin{aligned} M &= 2 \left[\ln(t + \sqrt{1+t^2}) \right]_{-1}^1 \\ &= 2 \ln(\sqrt{2} + 1) - 2 \ln(\sqrt{2} - 1) \simeq 3.53. \end{aligned}$$

So the mass of the rod is approximately 3.53 kilograms.

Exercise 3

A semicircular path C has parametric representation

$$x(t) = R \cos t, \quad y(t) = R \sin t \quad (0 \leq t \leq \pi),$$

where R is a positive constant. On this path, the linear number density of ants (i.e. the number of ants per unit length of path) is given by

$$\lambda(x, y) = \frac{A}{R^4} x^2 y,$$

where A is a positive constant. What is the total number of ants on the path?

1.2 Line integrals in orthogonal coordinates

Line integrals can also be evaluated using non-Cartesian coordinates. In this subsection, we describe how this is done. This is optional material, and will not be assessed. However, you should read it if you are interested in physics or astronomy, as there are close links with the mathematics used in Einstein's theory of relativity.

As an example, let us see how the length of a path is calculated in polar coordinates. We consider a path in the xy -plane that is defined using polar coordinates (r, ϕ) . This means that its parametric representation is given in the form

$$r = r(t), \quad \phi = \phi(t) \quad (t_1 \leq t \leq t_2).$$

Figure 11 shows the short segment between points P and Q . The point P has polar coordinates (r, ϕ) and parameter value t , and the point Q has polar coordinates $(r + \delta r, \phi + \delta \phi)$ and parameter value $t + \delta t$.

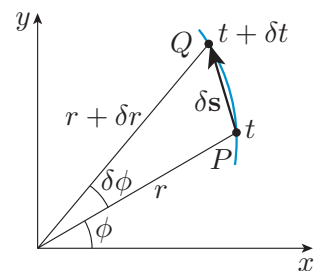


Figure 11 A path segment in polar coordinates (the angle $\delta\phi$ is magnified for clarity)

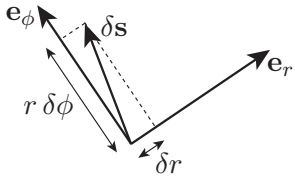


Figure 12 Resolving $\delta \mathbf{s}$ along polar unit vectors

The displacement vector $\delta \mathbf{s}$ from P to Q can be resolved into a radial component δr in the direction of the unit vector \mathbf{e}_r , and a transverse component $r \delta \phi$ in the direction of the unit vector \mathbf{e}_ϕ (see Figure 12). We therefore have

$$\delta \mathbf{s} = \delta r \mathbf{e}_r + r \delta \phi \mathbf{e}_\phi. \quad (10)$$

Note the factor r in the last term on the right-hand side. This is the *scale factor* needed to convert a change in angle, $\delta \phi$, into an appropriate length.

Bearing the scale factor in mind, we can follow the same argument as that given earlier for Cartesian coordinates. In polar coordinates, the expression for the length of a small segment of the path is

$$\delta l \simeq \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2} \delta t,$$

and the total length of path between points with parameter values $t = t_1$ and $t = t_2$ is

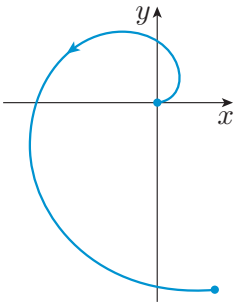
$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2} dt. \quad (11)$$

To illustrate this result, consider an anticlockwise circular path of radius R , centred on the origin. This may be described by the parametric equations

$$r(t) = R, \quad \phi(t) = t \quad (0 \leq t \leq 2\pi).$$

Since $dr/dt = 0$ and $d\phi/dt = 1$, the total length of the path is

$$L = \int_0^{2\pi} \sqrt{0^2 + R^2 \times 1} dt = \int_0^{2\pi} R dt = 2\pi R.$$



Exercise 4

In polar coordinates, the spiral path shown in the margin has parametric representation

$$r(t) = 2t, \quad \phi(t) = t \quad (0 \leq t \leq 5).$$

Find the length of this path.

(Hint: You may use the standard integral given in Exercise 1.)

The formula for the length of a path in polar coordinates can be generalised to other orthogonal coordinate systems.

Suppose that a path in three-dimensional space is described in orthogonal coordinates (u, v, w) . The corresponding scale factors are denoted by h_u , h_v and h_w , and the unit vectors by \mathbf{e}_u , \mathbf{e}_v and \mathbf{e}_w .

The path is then specified by parametric equations of the form

$$u = u(t), \quad v = v(t), \quad w = w(t) \quad (t_1 \leq t \leq t_2).$$

We consider two neighbouring points on the path: P with coordinates (u, v, w) and parameter value t , and Q with coordinates $(u + \delta u, v + \delta v, w + \delta w)$ and parameter value $t + \delta t$ (see Figure 13).

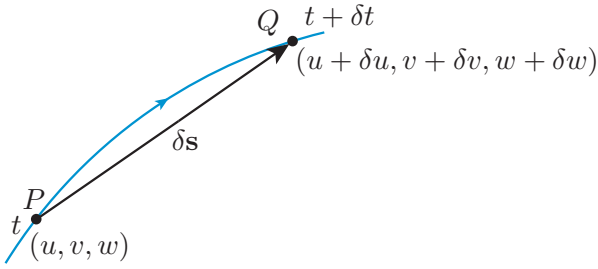


Figure 13 Neighbouring points P and Q on a path in (u, v, w) coordinates. Inevitably, this sketch is two-dimensional, but the path need not be planar.

The displacement vector $\delta \mathbf{s}$ from P to Q can be resolved in the directions of the \mathbf{e}_u , \mathbf{e}_v and \mathbf{e}_w unit vectors. For example, the component in the direction of \mathbf{e}_u is $h_u \delta u$. This follows directly from the definition of scale factors. There are similar results for the other components, so the displacement vector from P to Q is given by

$$\delta \mathbf{s} = h_u \delta u \mathbf{e}_u + h_v \delta v \mathbf{e}_v + h_w \delta w \mathbf{e}_w. \quad (12)$$

Using the same argument as before, we reach the following conclusion.

Length of a path in an orthogonal coordinate system

In an orthogonal coordinate system (u, v, w) , with scale factors h_u , h_v and h_w , the length of a path between points with parameter values $t = t_1$ and $t = t_2$ is

$$L = \int_{t_1}^{t_2} \sqrt{h_u^2 \left(\frac{du}{dt} \right)^2 + h_v^2 \left(\frac{dv}{dt} \right)^2 + h_w^2 \left(\frac{dw}{dt} \right)^2} dt. \quad (13)$$

The main point to note about this formula is the presence of the scale factors. These were not apparent in Cartesian coordinates (x, y, z) because all the scale factors are equal to 1 in this case. In polar coordinates (r, ϕ) , the scale factors are $h_r = 1$ and $h_\phi = r$, and there is no third coordinate, so in this special case we recover equation (11). For reference, some useful scale factors are listed in Table 1.

Table 1 Some scale factors

Coordinate system	Scale factors
Polar coordinates (r, ϕ)	$h_r = 1, h_\phi = r$
Cylindrical coordinates (r, ϕ, z)	$h_r = 1, h_\phi = r, h_z = 1$
Spherical coordinates (r, θ, ϕ)	$h_r = 1, h_\theta = r, h_\phi = r \sin \theta$

We can apply equation (13) to paths on the surface of a sphere, such as those that describe journeys on the surface of the Earth. It is natural to use spherical coordinates (r, θ, ϕ) in this case. The advantage of this choice is that the radial coordinate does not vary: it has the constant value R , the radius of the sphere.

The parametric equations of a path on the surface of the sphere then take the form

$$\theta = \theta(t), \quad \phi = \phi(t), \quad r = R \quad (t_1 \leq t \leq t_2).$$

Using equation (13), and taking scale factors from Table 1, the length of a path on the surface of a sphere is given by

$$L = R \int_{t_1}^{t_2} \sqrt{\left(\frac{d\theta}{dt}\right)^2 + \sin^2 \theta \left(\frac{d\phi}{dt}\right)^2} dt. \quad (14)$$

Numerical methods are often needed to evaluate this integral, but the cases considered in the following exercise can be done by hand.

Exercise 5

Paths A and B lie on the surface of a sphere of radius R , and have the following parametric representations.

$$\text{Path } A: \quad \theta(t) = t, \quad \phi = \pi/4 \quad (0 \leq t \leq \pi/2).$$

$$\text{Path } B: \quad \theta(t) = \pi/6, \quad \phi = t \quad (0 \leq t \leq \pi/2).$$

Use equation (14) to find the lengths of these paths.



Figure 14 A geodesic path from London to Los Angeles

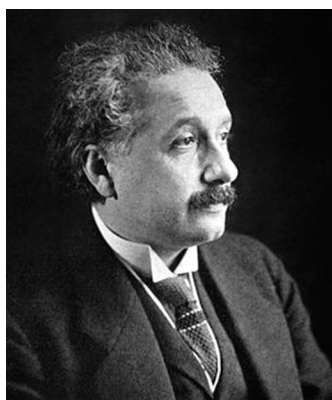


Figure 15 Albert Einstein (1882–1955)

Towards Einstein's theory of general relativity

When considering paths on a curved surface, it is natural to ask: what is the shortest path between two given points? Such paths are called **geodesics**. Long-distance plane routes generally follow geodesics, although slight adjustments may be made for winds and weather (Figure 14). The first step in finding geodesics is to have a formula for the length of a path, and this is provided by equation (13), although more work is needed to identify the geodesics. It turns out that in Exercise 5, path A is a geodesic but path B is not.

In 1916, after years of struggle, Albert Einstein (Figure 15) created his theory of **general relativity**. This is a theory of motion under gravity. Rather than dealing with ordinary space, general relativity deals with **spacetime**, which combines the three dimensions of space with time. Along the track of a particle in spacetime, a quantity called **proper time** increases steadily; this is analogous to the length of a curve in ordinary space.

General relativity is based on two extraordinary ideas. First, it asserts that spacetime is curved by matter. Then it says that when moving under gravity, a body follows the path of maximum proper time. By analogy with ordinary space, such a path is called a *geodesic*. So to predict motion under gravity, we must find the geodesics in spacetime, and this brings line integrals into the heart of general relativity.

2 Line integrals of vector fields

You have seen how to integrate *scalar fields* along given paths. It is also possible to integrate *vector fields* along paths, and this section explains how this is done.

2.1 The basic concept

A simple example will lead us towards the main idea. In a 100 metre race, competitors sprint along a straight track. When analysing the times achieved, officials often record the component of wind velocity in the direction of the race. Fast times are less impressive in a strong following wind, and world records cannot be claimed if such a wind is present.

Longer races generally follow curved paths. For distances above 400 metres, the runners complete one or more laps of the stadium. In these cases, it is not relevant to record the component of the wind velocity in any single direction. Assuming that the wind velocity field remains constant in time, a more suitable measure of wind assistance may be obtained as follows.

- At each point along the path of the race, measure the component of the wind velocity along the direction of the path.
- Integrate this component round the path, from its start point to its end point.

In races round complete laps, we might expect the wind to be in the runners' backs at some points, and in their faces at others. But we can also imagine situations where the wind swirls around the stadium like a gentle eddy, consistently helping the runners on their way. In practice, athletics officials do not concern themselves with such details, but this does not matter. The main point of our example is that it suggests a way of defining the line integral of a vector field.

The concept of the line integral of a vector field

Given a vector field \mathbf{v} , and a path C leading from a start point to an end point, the line integral of \mathbf{v} along C is defined as follows.

At each point along the path, we take the component of \mathbf{v} in the direction of the path, and then integrate this along the path.

As a simple example, consider the line integral of the vector field

$$\mathbf{v}(x, y) = x^2(y + 1)\mathbf{i} + x(y - 1)^2\mathbf{j}$$

along a path C that travels along the x -axis, starting at $x = 0$ and ending at $x = 3$. Figure 16 is an arrow map of this vector field, with the path C shown in blue.

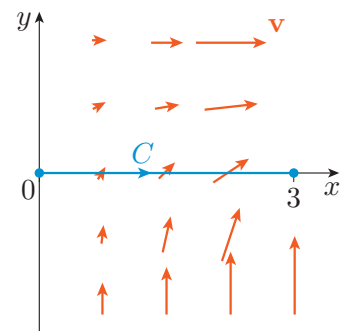


Figure 16 A vector field $\mathbf{v}(x, y)$ and a path C

In this case, the component of \mathbf{v} in the direction of the path is v_x . Along the path C , the component v_x has the value $x^2(0+1) = x^2$. So the line integral of the vector field \mathbf{v} along the path C is

$$\int_{x=0}^{x=3} x^2 dx = \left[\frac{1}{3}x^3 \right]_0^3 = 9.$$

Notice that the answer is a number. A similar result applies to all line integrals of vector fields – their values are scalars (i.e. numbers, or numbers with associated units).

This illustrates the concept of the line integral of a vector field in the special case where the path of integration is along a coordinate axis. But it does not give us a reliable way of calculating line integrals in general. More usually, the path of integration is curved, and the coordinates of points on the path are given by parametric equations. The next subsection will show you how to evaluate line integrals of vector fields in this general case. Once we have a suitable formula, the line integrals of vector fields are just as easy to evaluate as those of scalar fields.

2.2 Line integrals in Cartesian coordinates

Suppose that we are given a two-dimensional vector field $\mathbf{F}(x, y)$, and we want to calculate its line integral along a given path C , with start point A and end point B (Figure 17).

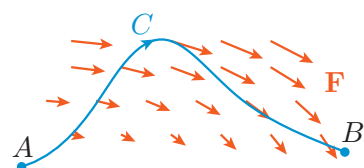


Figure 17 An arrow map of a vector field $\mathbf{F}(x, y)$ (orange) and a path C (blue)

We can imagine approximating the path C by a succession of straight-line steps, as shown in Figure 18. We do this by selecting points along the path, with position vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{n+1}$. The sequence starts from A , with position vector \mathbf{r}_1 , and ends at B , with position vector \mathbf{r}_{n+1} . We take a succession of straight-line steps: from \mathbf{r}_1 to \mathbf{r}_2 , from \mathbf{r}_2 to \mathbf{r}_3 , and so on, until we take the last step from \mathbf{r}_n to \mathbf{r}_{n+1} . The line integral of \mathbf{F} along the path C is approximated by a sum of contributions from all these steps.

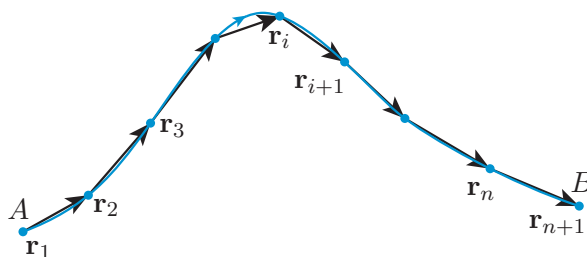


Figure 18 A path approximated by straight-line steps

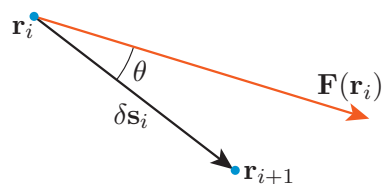


Figure 19 The i th step

Figure 19 takes a closer look at the step from \mathbf{r}_i to \mathbf{r}_{i+1} , which involves the displacement vector

$$\delta \mathbf{s}_i = \mathbf{r}_{i+1} - \mathbf{r}_i.$$

At the beginning of the step, the vector field has value $\mathbf{F}(\mathbf{r}_i)$. The contribution of this step to the complete line integral is given by multiplying the component of the field in the direction of the step

(which is $|\mathbf{F}(\mathbf{r}_i)| \cos \theta$) by the length of the step, $|\delta \mathbf{s}_i|$. Using the definition of the scalar product of two vectors, we therefore have

$$\text{contribution of } i\text{th step} = |\mathbf{F}(\mathbf{r}_i)| \cos \theta \times |\delta \mathbf{s}_i| = \mathbf{F}(\mathbf{r}_i) \cdot \delta \mathbf{s}_i,$$

Recall that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$.

where the last equality follows from the definition of the scalar product.

The value of the line integral along C can be closely approximated by adding similar contributions from all the steps:

$$\text{line integral} \simeq \sum_{i=1}^n \mathbf{F}(\mathbf{r}_i) \cdot \delta \mathbf{s}_i. \quad (15)$$

We consider this sum in the limit of an infinite number of infinitesimal steps. In this limit, any approximation involved in using a succession of straight-line steps disappears, and the sum gives the exact value of the line integral, which is written as

$$\text{line integral} = \int_C \mathbf{F} \cdot d\mathbf{s}. \quad (16)$$

Note that integral sign carries the label C , which indicates the path followed. It is *not* safe, in general, to write the line integral as

$$\int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{F} \cdot d\mathbf{s},$$

because this indicates only the start and end points, and makes no reference to the path taken between them. In general, we need to specify the *full* path before evaluating the line integral.

Using parametric representations

The concept of a line integral is straightforward, but there remains the task of evaluating equation (16) for a given vector field \mathbf{F} and a given path C . As for the line integrals of scalar functions, the key is to express the path in parametric form.

Let us suppose that the path C lies in the xy -plane, and that points on the path have the parametric representation

$$x = x(t), \quad y = y(t) \quad (t_1 \leq t \leq t_2),$$

where, as usual, $t = t_1$ refers to the start point of the path and $t = t_2$ refers to the end point. We consider the small displacement $\delta \mathbf{s}$ produced when we move from a point P with parameter value t to a neighbouring point Q with parameter value $t + \delta t$ (see Figure 20).

An expression for this displacement was obtained in equation (3),

$$\delta \mathbf{s} = \delta x \mathbf{i} + \delta y \mathbf{j},$$

so

$$\begin{aligned} \mathbf{F} \cdot \delta \mathbf{s} &= (F_x \mathbf{i} + F_y \mathbf{j}) \cdot (\delta x \mathbf{i} + \delta y \mathbf{j}) \\ &= F_x \delta x + F_y \delta y. \end{aligned}$$

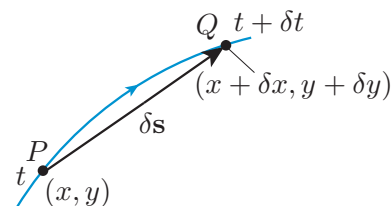


Figure 20 The small displacement from P to Q

Dividing and multiplying by δt then gives

$$\begin{aligned}\mathbf{F} \cdot \delta \mathbf{s} &= \mathbf{F} \cdot \frac{\delta \mathbf{s}}{\delta t} \delta t \\ &\simeq \mathbf{F} \cdot \frac{d\mathbf{s}}{dt} \delta t \\ &= \left(F_x \frac{dx}{dt} + F_y \frac{dy}{dt} \right) \delta t.\end{aligned}$$

To obtain the line integral, we take the sum of expressions like this all along the path in the limit of an infinite number of infinitesimal steps. This gives the following result.

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{s} &= \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{d\mathbf{s}}{dt} dt \\ &= \int_{t_1}^{t_2} \left(F_x \frac{dx}{dt} + F_y \frac{dy}{dt} \right) dt.\end{aligned}\tag{17}$$

This formula is very important. It expresses the line integral of a vector field as a definite integral over the parameter t . To evaluate the definite integral, we must express the integrand on the right-hand side as a function of t . The following example shows how this is done.

Example 3

Calculate the line integral of the vector field

$$\mathbf{F} = (x - y)\mathbf{i} + (x + y)\mathbf{j}$$

along the quarter-circle path C_1 in Figure 21.

This path can be parametrised by the equations

$$x = 2 \cos t, \quad y = 2 \sin t \quad (0 \leq t \leq \pi/2).$$

Solution

Differentiating the parametric equations gives

$$\frac{dx}{dt} = -2 \sin t, \quad \frac{dy}{dt} = 2 \cos t.$$

Expressing the components of \mathbf{F} in terms of t gives

$$\begin{aligned}F_x &= x - y = 2(\cos t - \sin t), \\ F_y &= x + y = 2(\cos t + \sin t).\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{F} \cdot \frac{d\mathbf{s}}{dt} &= F_x \frac{dx}{dt} + F_y \frac{dy}{dt} \\ &= -4(\cos t - \sin t) \sin t + 4(\cos t + \sin t) \cos t \\ &= 4(\sin^2 t + \cos^2 t) \\ &= 4.\end{aligned}$$

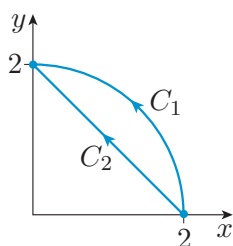


Figure 21 Two paths between the same points

Substituting into equation (17), the required line integral is

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_0^{\pi/2} 4 \, dt = [4t]_0^{\pi/2} = 2\pi.$$

Line integrals generally depend on the path between their start and end points. For example, the line integral of \mathbf{F} over the path C_2 in Figure 21 has a different value to that calculated in Example 3, as you can confirm.

Exercise 6

Calculate the line integral of the vector field

$$\mathbf{F} = (x - y)\mathbf{i} + (x + y)\mathbf{j}$$

along the straight-line path C_2 shown in Figure 21. This path can be parametrised by the equations

$$x = 2 - t, \quad y = t \quad (0 \leq t \leq 2).$$

The method used in the above example and exercise applies to all line integrals of vector fields. In three dimensions, the parametric equations also include a function $z(t)$, and the expression for $d\mathbf{s}/dt$ has a component dz/dt . The following procedure refers to this three-dimensional case, but it is readily adapted to curves in the xy -plane by omitting the terms involving z .

Procedure 1 Finding the line integral of a vector field

Given a vector field $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$, and a path C with the parametric representation

$$x = x(t), \quad y = y(t), \quad z = z(t) \quad (t_1 \leq t \leq t_2),$$

the line integral of \mathbf{F} along the path C can be found as follows.

1. Use the parametric representation to find the components of

$$\frac{d\mathbf{s}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}. \quad (18)$$

2. Express the components of \mathbf{F} as functions of the parameter t .
3. Find the scalar product

$$\mathbf{F} \cdot \frac{d\mathbf{s}}{dt} = F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt}$$

as a function of t .

4. Evaluate the line integral as a definite integral over t :

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{d\mathbf{s}}{dt} dt. \quad (19)$$

Example 4

Evaluate the line integral of the vector field

$$\mathbf{F} = x^2 \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k}$$

along a path C with the parametric equations

$$x = 3t^2, \quad y = 4t^2, \quad z = 5t^2 \quad (0 \leq t \leq 1).$$

Solution

Differentiating the parametric equations, we get

$$\frac{dx}{dt} = 6t, \quad \frac{dy}{dt} = 8t, \quad \frac{dz}{dt} = 10t.$$

Expressing the components of \mathbf{F} in terms of t gives

$$F_x = x^2 = 9t^4, \quad F_y = y^2 = 16t^4, \quad F_z = z^2 = 25t^4.$$

So

$$\begin{aligned} \mathbf{F} \cdot \frac{d\mathbf{s}}{dt} &= 9t^4 \times 6t + 16t^4 \times 8t + 25t^4 \times 10t \\ &= 432t^5. \end{aligned}$$

Substituting into equation (19), the line integral is

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 432t^5 dt \\ &= 432 \left[\frac{1}{6}t^6 \right]_0^1 = 72. \end{aligned}$$

Exercise 7

Calculate the line integral of the vector field

$$\mathbf{F} = yz \mathbf{i} + xz \mathbf{j} + xy \mathbf{k}$$

along a path C with parametric equations

$$x = t, \quad y = 1 + 2t, \quad z = 4 \quad (0 \leq t \leq 1).$$

This is different to the behaviour of the lengths of curves and the line integrals of scalar functions, which are unchanged by a reversal of the path.

If we reverse the direction of the path of a line integral of a vector field, tracing out the same curve but in the opposite sense, the magnitude of the line integral remains unchanged, but its sign is reversed.

This follows directly from equation (15), which expresses the line integral of a vector field \mathbf{F} as a sum of contributions of the form $\mathbf{F}(\mathbf{r}_i) \cdot \delta\mathbf{s}_i$. When we reverse the direction of the path, the sign of each small displacement $\delta\mathbf{s}_i$ changes, so the contribution changes to

$$\mathbf{F}(\mathbf{r}_i) \cdot (-\delta\mathbf{s}_i) = -\mathbf{F}(\mathbf{r}_i) \cdot \delta\mathbf{s}_i.$$

Since this is true for all contributions along the path, the line integral along the reverse path is *minus* the line interval along the original path.

You can check this in the following exercise.

Exercise 8

Calculate the line integral of the vector field

$$\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

along the path C_{rev} , with parametric equations

$$x = 1 - t, \quad y = 3 - 2t, \quad z = 4 \quad (0 \leq t \leq 1).$$

This is the reverse of the path C in Exercise 7, obtained by replacing t with $1 - t$ in the parametric equations.

The following optional box makes a link between line integrals and the concept of energy; this link is of fundamental importance in physics.

Line integrals and energy

Think of a particle of mass m , moving along a path C . For example, it could be a ball that has been thrown in the air, and follows a parabolic arc back to the ground (Figure 22). At each point \mathbf{r} , the particle experiences a force $\mathbf{F}(\mathbf{r})$, and we can calculate the line integral of $\mathbf{F}(\mathbf{r})$ along the particle's path:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{d\mathbf{s}}{dt} dt, \quad (20)$$

where t is the parameter used to label points along the path, with $t = t_1$ at the start point, and $t = t_2$ at the end point.

The parameter t need not have any physical significance. However, we are free to choose it to be the time elapsed, which increases as the particle traces out its path. This choice does not affect the value of the line integral, but helps us to interpret its meaning.

With t interpreted as time, $d\mathbf{s}/dt$ is the velocity \mathbf{v} of the particle. Also, Newton's second law tells us that 'force is equal to mass times acceleration'. Since acceleration is the rate of change of velocity, we can express this as $\mathbf{F} = m d\mathbf{v}/dt$. Putting these results together, the integrand on the right-hand side of equation (20) is

$$\begin{aligned} \mathbf{F} \cdot \frac{d\mathbf{s}}{dt} &= m \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} \\ &= m \left(\frac{dv_x}{dt} v_x + \frac{dv_y}{dt} v_y + \frac{dv_z}{dt} v_z \right) \\ &= m \frac{d}{dt} \left[\frac{1}{2} (v_x^2 + v_y^2 + v_z^2) \right] \\ &= \frac{d}{dt} \left(\frac{1}{2} m v^2 \right), \end{aligned} \quad (21)$$

where $v = |\mathbf{v}|$ is the speed of the particle, which is a function of time as the particle progresses along its path.

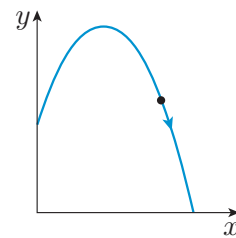


Figure 22 A thrown ball follows a parabolic arc

Combining equations (20) and (21), we conclude that

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{1}{2}mv^2 \right) dt = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2, \quad (22)$$

where $v_1 = v(t_1)$ is the particle's initial speed at the beginning of the path, and $v_2 = v(t_2)$ is its final speed at the end.

Scientists place a special interpretation on these results. The quantity $\frac{1}{2}mv^2$ is called the **kinetic energy** of the particle, and is interpreted as the energy that the particle has by virtue of being in motion. If $v_2 > v_1$ in equation (22), the particle gains kinetic energy as it moves along the path C .

It is a fundamental principle of science that energy is conserved, so the energy gained by the particle must come from somewhere. If you push the particle in the absence of other forces, it comes from energy stored in your muscles. If the particle falls under gravity, it comes from a type of energy known as **potential energy**, which is stored in the gravitational field. You need not worry about the details – the important point is that the line integral on the left-hand side of equation (22) quantifies the energy transferred to the particle by the force \mathbf{F} . This is a major application of line integrals.

The final conclusion does not depend on the use of time as the parameter in the line integral.

We note in passing that it is also possible to evaluate the line integrals of vector fields in non-Cartesian coordinate systems. If (u, v, w) are orthogonal coordinates with unit vectors \mathbf{e}_u , \mathbf{e}_v and \mathbf{e}_w , then a vector field \mathbf{F} may be expressed as

$$\mathbf{F} = F_u \mathbf{e}_u + F_v \mathbf{e}_v + F_w \mathbf{e}_w.$$

In terms of these coordinates, a path C is represented by parametric equations of the form

$$u = u(t), \quad v = v(t), \quad w = w(t) \quad (t_1 \leq t \leq t_2),$$

and the line integral of \mathbf{F} along C is given by the general formula

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{t_1}^{t_2} \left(F_u h_u \frac{du}{dt} + F_v h_v \frac{dv}{dt} + F_w h_w \frac{dw}{dt} \right) dt, \quad (23)$$

where h_u , h_v and h_w are the appropriate scale factors. In the special case of Cartesian coordinates, all these scale factors are equal to 1, and we recover equation (18). You will not be asked to use equation (23) in this module, but we quote it to give you a more complete picture, and because you may come across it in your future studies.

3 Line integrals of gradient fields

You have seen how to calculate the line integral of any vector field. This section considers a restricted class of vector fields – those that are proportional to the gradients of scalar fields. The line integrals of these fields have a special property: *they are independent of the path taken between their start and end points*. This property is very useful in applications, including many that arise in physics and engineering.

3.1 Gradient fields

We often meet vector fields that are expressed in the form

$$\mathbf{F} = -\nabla U,$$

where U is a scalar field. The minus sign in this equation means that the vector field \mathbf{F} points in the direction in which the scalar field U *decreases* most rapidly. This is usually what is needed. For example, heat flows in the direction in which temperature decreases most rapidly. We retain the minus sign throughout our discussion because this is what you are most likely to meet in real-world applications beyond this module.

A vector field \mathbf{F} is called a **gradient field** if it can be expressed in the form

$$\mathbf{F} = -\nabla U, \quad (24)$$

where U is a scalar field.

U is called the **scalar potential field** associated with \mathbf{F} .

Not all vector fields are gradient fields, and a test for gradient fields will be given later in this section.

Example 5

Show that the vector field $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$ is a gradient field with an associated scalar potential field $U = \frac{1}{2}(y^2 - x^2)$.

Solution

Taking the partial derivatives of $U = \frac{1}{2}(y^2 - x^2)$, we get

$$\frac{\partial U}{\partial x} = -x \quad \text{and} \quad \frac{\partial U}{\partial y} = y,$$

so

$$\nabla U = \frac{\partial U}{\partial x} \mathbf{i} + \frac{\partial U}{\partial y} \mathbf{j} = -x \mathbf{i} + y \mathbf{j} = -\mathbf{F}.$$

Gradient fields have a very important property: their line integrals depend on the start and end points of the path, but are independent of the detailed shape of the path joining these points.

To see why this is so, let us consider a gradient field

$$\mathbf{F} = -\nabla U.$$

We can take the line integral of this vector field along a path C with start point A and end point B . Points on this path are labelled by a parameter t , with $t = t_A$ at the start point, and $t = t_B$ at the end point. The line integral is given by

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{t_A}^{t_B} \mathbf{F} \cdot \frac{d\mathbf{s}}{dt} dt = - \int_{t_A}^{t_B} \nabla U \cdot \frac{d\mathbf{s}}{dt} dt. \quad (25)$$

Now, the integrand on the right-hand side can be simplified:

$$\begin{aligned} \nabla U \cdot \frac{d\mathbf{s}}{dt} &= \left(\frac{\partial U}{\partial x} \mathbf{i} + \frac{\partial U}{\partial y} \mathbf{j} + \frac{\partial U}{\partial z} \mathbf{k} \right) \cdot \left(\frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \right) \\ &= \frac{\partial U}{\partial x} \frac{dx}{dt} + \frac{\partial U}{\partial y} \frac{dy}{dt} + \frac{\partial U}{\partial z} \frac{dz}{dt} = \frac{dU}{dt}, \end{aligned}$$

where the last step uses a version of the chain rule (equation (23) of Unit 7). Using this result in the integral on the far right-hand side of equation (25), we get

$$\int_C \mathbf{F} \cdot d\mathbf{s} = - \int_{t_A}^{t_B} \frac{dU}{dt} dt = U_A - U_B, \quad (26)$$

where $U_A = U(t_A)$ is the value of U at the start point of the path, and $U_B = U(t_B)$ is the value of U at the end point.

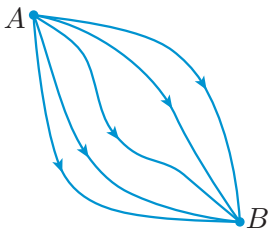


Figure 23 Paths from A to B

The same answer $U_A - U_B$ is obtained no matter which route is taken between the given start and end points. For example, Figure 23 shows several paths leading from A to B . The line integral of a gradient field $\mathbf{F} = -\nabla U$ has the same value for all these paths. For a closed loop, the start point A is the same as the end point B , so $U_A = U_B$. Hence the line integral of a gradient field around a closed loop is equal to zero.

A line integral that does not depend on the path between given start and end points is said to be **path-independent**. We can therefore make the following statements.

- Any line integral of a gradient field is *path-independent*.
- Any line integral of a gradient field around a *closed loop* is equal to zero.

It is easy to find the line integral of a gradient field if we know its associated scalar potential field, as the following example shows.

Example 6

The vector field $\mathbf{F} = 3x^2y \mathbf{i} + (x^3 + 4y^3) \mathbf{j}$ is a gradient field with associated scalar potential field $U(x, y) = 1 - x^3y - y^4$. Find the line integral of \mathbf{F} along a path C with parametric equations

$$x = 1 - \cos t, \quad y = 1 + \sin t \quad (0 \leq t \leq \pi).$$

Solution

Because \mathbf{F} is a gradient field, its line integrals are path-independent. Although the path is specified in detail, only its start and end points matter. Substituting $t = 0$ and $t = \pi$ in the parametric equations gives start point $(0, 1)$ and end point $(2, 1)$. Using equation (26), we then get

$$\int_C \mathbf{F} \cdot d\mathbf{s} = U(0, 1) - U(2, 1) = 0 - (1 - 8 - 1) = 8.$$

Exercise 9

The vector field $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$ is a gradient field with an associated scalar potential field $U = \frac{1}{2}(y^2 - x^2)$. Find the line integral of \mathbf{F} along any path C that starts from $(1, 1)$ and ends at $(7, 3)$.

3.2 Conservative vector fields

You have seen that the line integrals of gradient fields are always *path-independent*, and this simplifies the evaluation of these line integrals. In order to focus on the property of path-independence, we make a definition.

Conservative fields

A vector field \mathbf{F} is said to be **conservative** if, throughout its domain, all of its line integrals are path-independent.

Using this definition, it is clear that all gradient fields are conservative fields. But can we say that all conservative fields are gradient fields? Our definitions do not exclude the possibility that a field could be conservative, and yet not be expressible in the form $\mathbf{F} = -\nabla U$. You will soon see that all conservative fields *are* gradient fields, but a little more work is needed to establish this fact.

Why conservative?

The word ‘conservative’ is used for historical reasons. An early application of line integrals was to calculate the energy transferred by a force when a particle moves from one point to another. If these line integrals are path-independent, it turns out that the law of conservation of energy can be expressed by a simple formula involving kinetic and potential energies. The term *conservative field* derives from *conservation of energy*, but is now a far more general concept.

If we are told that a given vector field is conservative, we can often simplify the evaluation of its line integrals, as the following example shows.

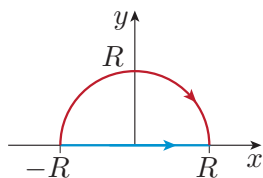


Figure 24 A semicircular path and its straight-line replacement

Example 7

The vector field $\mathbf{F} = (x^2 + y^2)\mathbf{i} + 2xy\mathbf{j}$ is conservative. Calculate its line integral along the red semicircular path in Figure 24.

Solution

We are told that \mathbf{F} is a conservative field, so we are free to replace the semicircular path by a straight-line path along the x -axis, shown in blue in Figure 24. Along this straight-line path, the component of \mathbf{F} in the direction of the path is $F_x = x^2 + 0 = x^2$, so the line integral is

$$\int_{-R}^R F_x dx = \int_{-R}^R x^2 dx = \left[\frac{1}{3}x^3\right]_{-R}^R = \frac{2}{3}R^3.$$

This is equal to the required line integral because the blue and red paths share the same start and end points, and the field is conservative.

Exercise 10

The vector field $\mathbf{F} = 3x^2y\mathbf{i} + (x^3 + y^3)\mathbf{j}$ is conservative. Find the line integral of \mathbf{F} along any path that starts at the origin and ends at $(1, 2)$.

But be very careful: you *cannot* adjust the path of a line integral of a vector field *unless* you know that the field is conservative. If the field is not conservative, changing the path will generally give the wrong answer!

We can use an alternative notation for the line integrals of conservative fields, reflecting the fact that they are path-independent. The line integral of the conservative field \mathbf{F} along a path C with start point A and end point B can be written in any of the forms

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{A \rightarrow B} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{r}_A \rightarrow \mathbf{r}_B} \mathbf{F} \cdot d\mathbf{s},$$

where \mathbf{r}_A and \mathbf{r}_B are the position vectors of A and B . The notations used on the right have the advantage of explicitly indicating the start and end points without giving *irrelevant* information about the precise shape of the path. Using this notation, and referring to Figure 25, it is easy to see that for any *conservative* vector field \mathbf{F} ,

$$\int_{\mathbf{r}_0 \rightarrow \mathbf{r}_B} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{r}_0 \rightarrow \mathbf{r}_A} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{r}_A \rightarrow \mathbf{r}_B} \mathbf{F} \cdot d\mathbf{s}. \quad (27)$$

This expresses the fact that the line integral from \mathbf{r}_0 to \mathbf{r}_B is the same whether the route goes via \mathbf{r}_A or not.

We now show that any conservative field \mathbf{F} must also be a gradient field. We do this by constructing a scalar function $U(\mathbf{r})$ for which $\mathbf{F} = -\nabla U$. This is done by first choosing a fixed reference point \mathbf{r}_0 at which $U(\mathbf{r}_0) = 0$, and then defining

$$U(\mathbf{r}) = - \int_{\mathbf{r}_0 \rightarrow \mathbf{r}} \mathbf{F} \cdot d\mathbf{s}. \quad (28)$$

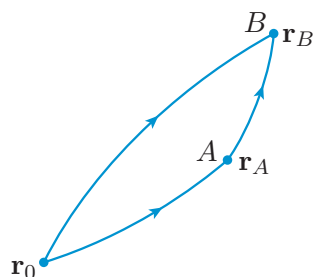


Figure 25 Two routes from \mathbf{r}_0 to \mathbf{r}_B

The choice of \mathbf{r}_0 is arbitrary, but this does not matter – we just make some definite choice.

For a conservative vector field \mathbf{F} and a fixed reference point \mathbf{r}_0 , the line integral on the right-hand side depends only on the end point \mathbf{r} . So $U(\mathbf{r})$ is a well-defined function, which is equal to zero at $\mathbf{r} = \mathbf{r}_0$. Because the line integral involves a scalar product, $U(\mathbf{r})$ is a scalar quantity – in fact, it is the *scalar potential field* associated with \mathbf{F} , as we will now show.

When we move from a point \mathbf{r}_A to another point \mathbf{r}_B , equation (28) tells us that U changes by

$$U(\mathbf{r}_B) - U(\mathbf{r}_A) = - \int_{\mathbf{r}_0 \rightarrow \mathbf{r}_B} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{r}_0 \rightarrow \mathbf{r}_A} \mathbf{F} \cdot d\mathbf{s}.$$

Using equation (27), this is equivalent to

$$U(\mathbf{r}_B) - U(\mathbf{r}_A) = - \int_{\mathbf{r}_A \rightarrow \mathbf{r}_B} \mathbf{F} \cdot d\mathbf{s}. \quad (29)$$

Now let us take $\mathbf{r}_A = (x, y, z)$ and $\mathbf{r}_B = (x + \delta x, y, z)$, where δx is a tiny increment in x . Then the line integral on the right-hand side of equation (29) can be taken parallel to the x -axis. If δx is very small, we get

$$U(x + \delta x, y, z) - U(x, y, z) \simeq -F_x(x, y, z) \delta x.$$

Then dividing both sides by δx and taking the limit as δx tends to zero, we get

$$\frac{\partial U}{\partial x} = -F_x.$$

Similar results for $\partial U / \partial y$ and $\partial U / \partial z$ are obtained by making tiny displacements in the y - and z -directions. Collecting these results together, we conclude that

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z},$$

or in vector form,

$$\mathbf{F} = -\nabla U.$$

This shows that any conservative field is a gradient field. We already know that any gradient field is a conservative field, so we reach the following memorable and important conclusion.

The terms *conservative field* and *gradient field* are synonymous and can be used interchangeably.

Given a conservative field \mathbf{F} , we often want to find the corresponding scalar potential field U . This is useful because once we know U , we can easily evaluate any line integral of \mathbf{F} using equation (29). To obtain a formula for U , we can use equation (28) with any convenient choice of path for the line integral. This technique is illustrated in the following example.

For a point $\mathbf{r} = (x, y, z)$, we use the notations $U(\mathbf{r})$ and $U(x, y, z)$ interchangeably.

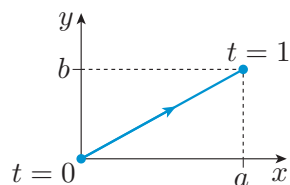


Figure 26 A straight-line path from the origin to (a, b)

Example 8

A conservative vector field takes the form $\mathbf{F} = xy^2 \mathbf{i} + x^2y \mathbf{j}$. Find the associated scalar potential field $U(x, y)$, taking $U = 0$ at the origin $\mathbf{0}$.

Solution

From equation (28), the scalar potential field is given by

$$U(\mathbf{r}) = - \int_{\mathbf{0} \rightarrow \mathbf{r}} \mathbf{F} \cdot d\mathbf{s}.$$

Because the vector field is conservative, we can evaluate this line integral over any convenient path. Let us consider an arbitrary point (a, b) . We choose a straight-line path from the origin to this point, as shown in Figure 26.

This path can be described by the parametric equations

$$x = at, \quad y = bt \quad (0 \leq t \leq 1).$$

The values of a and b are constant along the path, so

$$\frac{dx}{dt} = a \quad \text{and} \quad \frac{dy}{dt} = b.$$

Hence

$$\mathbf{F} \cdot \frac{d\mathbf{s}}{dt} = (at)(bt)^2a + (at)^2(bt)b = 2a^2b^2t^3$$

and

$$U(a, b) = - \int_{t=0}^{t=1} 2a^2b^2t^3 dt = -\frac{1}{2}a^2b^2.$$

However, the point (a, b) is arbitrary, so for any point (x, y) , we conclude that the scalar potential field is

$$U(x, y) = -\frac{1}{2}x^2y^2.$$

This answer can (and should) be checked by taking its gradient:

$$\nabla U = -xy^2 \mathbf{i} - x^2y \mathbf{j} = -\mathbf{F},$$

as required.

Exercise 11

A conservative vector field takes the form $\mathbf{F} = \cos x \mathbf{i} + \sin y \mathbf{j}$. Find the associated scalar potential field $U(x, y)$, taking $U(0, 0) = 0$.

3.3 The curl test

This subsection gives a test that allows us to decide whether or not a given vector field is conservative. We start by noting that if \mathbf{F} is a conservative vector field, then it is also a gradient field and can be written in the form

$$\mathbf{F} = -\nabla U.$$

Taking the curl of both sides of this equation, we get

$$\nabla \times \mathbf{F} = -\nabla \times (\nabla U) = - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{vmatrix}.$$

Expanding the determinant on the right-hand side in the usual way, and using the mixed partial derivative theorem of Unit 7, we get the zero vector $\mathbf{0} = (0, 0, 0)$. For example, the x -component is

$$-\frac{\partial}{\partial y} \left(\frac{\partial U}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial U}{\partial y} \right) = 0.$$

So any conservative field has zero curl throughout its domain.

In most circumstances, the converse statement is also true; so if a vector field \mathbf{F} has $\nabla \times \mathbf{F} = \mathbf{0}$ throughout its domain, then \mathbf{F} is conservative. We do not have the tools to prove this yet, but a proof is given at the end of the unit. Taking the result on trust, and assuming that all the necessary conditions are met, leads to the following *curl test*. This is the normal way of deciding whether or not a given vector field is conservative.

This calculation was done in Exercise 27 of Unit 9.

Curl test for conservative fields

To test whether the vector field \mathbf{F} is conservative, evaluate $\nabla \times \mathbf{F}$.

If $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere in the domain of \mathbf{F} , then \mathbf{F} is conservative. Otherwise, it is not conservative.

Strictly speaking, the curl test assumes that the domain of \mathbf{F} is ‘simple’ in a certain sense (technically, it must be *simply-connected*). We will return to this point in Subsection 5.3.

Example 9

Determine whether or not the following vector fields are conservative.

- (a) $\mathbf{F} = yz \mathbf{i} + xz \mathbf{j} + xy \mathbf{k}$ (b) $\mathbf{G} = z^2 \mathbf{i} + x^2 \mathbf{j} + y^2 \mathbf{k}$

Solution

- (a) We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \mathbf{i}(x - x) - \mathbf{j}(y - y) + \mathbf{k}(z - z) = \mathbf{0}.$$

So the curl test shows that \mathbf{F} is conservative.

- (b) We have

$$\nabla \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & y^2 \end{vmatrix} = \mathbf{i}(2y) - \mathbf{j}(-2z) + \mathbf{k}(2x).$$

This is not equal to $\mathbf{0}$ everywhere, so \mathbf{G} is *not* conservative.

Exercise 12

Determine whether or not the following vector fields are conservative.

- (a) $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ (b) $\mathbf{G} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$

4 Flux and the divergence theorem

This section introduces the concept of the *flux of a vector field*, which is a type of surface integral. Unit 8 showed you how to integrate scalar functions over surfaces. Here, we explain how to calculate surface integrals of vector functions. These integrals are found throughout science and engineering. For example, they are used to calculate the amount of air flowing out of a given region or the rate at which heat energy is lost through the walls, roof and ground floor of a house. They are also important in electromagnetism.

The concept of flux allows us to quantify the extent to which a vector field diverges outwards from a given point. Once this has been understood, we can return to a major theme of Unit 9 – *divergence* and its interpretation. An important result called the *divergence theorem* will cast further light on the meaning of divergence.

4.1 Flux over a planar surface element

You probably think of area as a scalar quantity – a certain number of square metres or square inches, say. This is fine for areas drawn on a sheet of paper. More generally, we can consider a planar element that is oriented in three-dimensional space. Such an element is characterised by its area δS and its orientation in space.

The simplest way of describing the orientation is to specify a unit vector $\hat{\mathbf{n}}$ that is perpendicular to the surface of the element (Figure 27). There are actually two vectors that could be used for this purpose, pointing in opposite directions. This is not a problem – we just pick one of these vectors and specify our selection clearly. The chosen unit vector is then called the **unit normal** of the planar element.

Information about the area of the element and its orientation can be combined. We define the **oriented area** of the element to be

$$\delta\mathbf{S} = \delta S \hat{\mathbf{n}}. \quad (30)$$

This is a vector quantity. Its magnitude is the (scalar) area of the element, δS , and its direction gives the direction of the unit normal to the element. Figure 28 shows the oriented area vectors of some planar elements. The magnitudes of these vectors are larger for elements of larger area, and this is indicated by the relative lengths of the arrows.

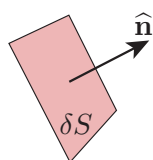


Figure 27 A planar element and its unit normal $\hat{\mathbf{n}}$

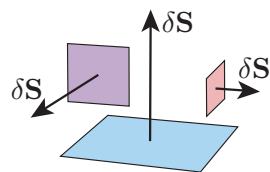


Figure 28 Some oriented areas

Now consider a fluid such as water or air moving in three-dimensional space. We can imagine a tiny planar element inside this fluid. This element is a mathematical construction rather than a tangible object, so it does not interfere with the flow of the fluid. We can then ask: how much fluid passes through the planar element per unit time?

In the simplest case, the planar element is perpendicular to the flow of the fluid, with its unit normal $\hat{\mathbf{n}}$ in the same direction as the fluid flow, as in Figure 29(a). If the fluid has velocity \mathbf{v} and speed $v = |\mathbf{v}|$, then the volume of fluid passing through the element in a small time δt is equal to the volume of the red box in Figure 29(a). This box has length $v \delta t$ and cross-sectional area δS , so the volume of fluid passing through the element in time δt is

$$\delta V = v \delta t \delta S.$$

Now let us see what happens when the planar element is *not* perpendicular to the flow. Figure 29(b) shows a case where the unit normal $\hat{\mathbf{n}}$ makes an angle θ with the velocity vector \mathbf{v} of the fluid. In this general case, the volume of fluid passing through the element in time δt is equal to the volume of the red skewed box in Figure 29(b).

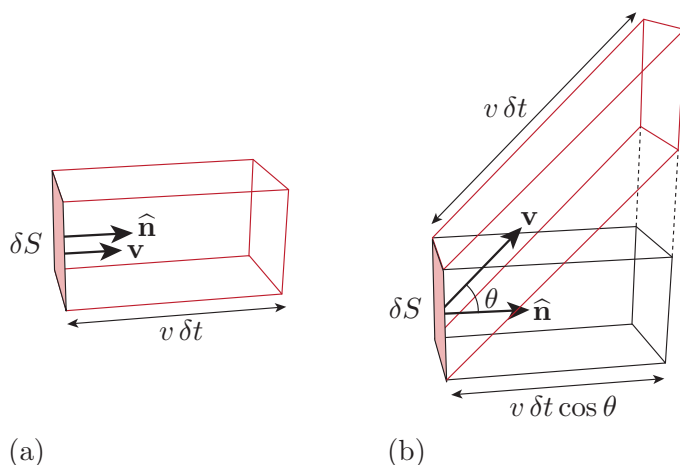


Figure 29 Flow through planar elements that are: (a) perpendicular to the flow; (b) not perpendicular to the flow

However, geometry tells us that the red skewed box has the same volume as the black box in Figure 29(b), so its volume is

$$\delta V = (v \delta t \cos \theta) \delta S = (v \cos \theta) \delta t \delta S.$$

The quantity $v \cos \theta$ that appears in this equation is the component of the fluid velocity in the direction of the unit normal, since

$$\hat{\mathbf{n}} \cdot \mathbf{v} = |\hat{\mathbf{n}}| |\mathbf{v}| \cos \theta = v \cos \theta.$$

Hence

$$\delta V = (\hat{\mathbf{n}} \cdot \mathbf{v}) \delta t \delta S.$$

Using the definition of the oriented area of a planar element in equation (30), this can also be written as

$$\delta V = (\mathbf{v} \cdot \delta \mathbf{S}) \delta t.$$

Dividing both sides by δt , and taking the limit as δt tends to zero, we see that the rate of flow of fluid volume through the planar element is

$$\frac{dV}{dt} = \mathbf{v} \cdot \delta \mathbf{S}. \quad (31)$$

The right-hand side of equation (31) is what we have been building towards. The quantity $\mathbf{v} \cdot \delta \mathbf{S}$ is called the *flux* of \mathbf{v} over the planar element with oriented area $\delta \mathbf{S}$, and we have shown that this is equal to the rate of flow of fluid through the element.

More generally, we can define the flux of *any* vector field.

Flux of a vector field over a planar element

Given any vector field \mathbf{F} and a planar element with oriented area $\delta \mathbf{S}$, the **flux** of the vector field over the element is defined as

$$\text{flux} = \mathbf{F} \cdot \delta \mathbf{S} = F \delta S \cos \theta, \quad (32)$$

where the field \mathbf{F} is evaluated at the position of the element, and θ is the angle between the directions of \mathbf{F} and the unit normal $\hat{\mathbf{n}}$ to the planar element.

We have

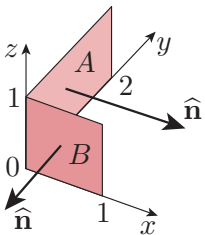
$$\mathbf{F} \cdot \delta \mathbf{S} = (\hat{\mathbf{n}} \cdot \mathbf{F}) \delta S.$$

Since $\hat{\mathbf{n}} \cdot \mathbf{F}$ is the normal component of the field, we can also state the following.

The flux of a vector field over a planar element is the *normal component* of the field multiplied by the area of the element.

Flux is a scalar quantity, which can be positive, negative or zero depending on the relative orientations of \mathbf{F} and the unit normal $\hat{\mathbf{n}}$.

The word ‘flux’ derives from a Latin word for flow, and you have seen that it allows us to describe the rate of flow of a fluid through a planar element. However, equation (32) defines flux for *any* vector field. When describing electric and magnetic fields, for example, we may be interested in their fluxes, even though these fields do not actually flow.



Exercise 13

The figure in the margin shows two planar elements A and B , with their unit normals indicated. Find the flux of the vector field $\mathbf{F} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ over both of these elements.

4.2 Flux over an extended surface

In general, we are interested in the flux of a vector field over an *extended surface*, which may be curved. This can be found by integrating over the surface, as we now explain.

Figure 30 shows a vector field \mathbf{F} and an extended curved surface S . The vector field, and its angle to the normal to the surface, may vary from point to point. However, we can imagine dividing the surface into many tiny elements, each of which can be approximated by a tiny planar area element, with oriented area $\delta\mathbf{S}_i$, at position \mathbf{r}_i . For each planar element, we have a choice of two directions for the unit normal. We require these choices to be made consistently so that neighbouring elements have unit normals that are nearly parallel rather than nearly antiparallel.

The flux of \mathbf{F} over the extended surface S is then approximated by

$$\text{flux over } S \simeq \sum_i \mathbf{F}(\mathbf{r}_i) \cdot \delta\mathbf{S}_i,$$

where the sum is over all the planar elements approximating the surface. We take the limit where the size of each element tends to zero and the number of elements tends to infinity. In this limit, the sum is written as

$$\text{flux over } S = \int_S \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S}, \quad (33)$$

where the expression on the right-hand side is called the **surface integral** of \mathbf{F} over the surface S . We will explain how to evaluate this surface integral shortly.

First, it is important to distinguish between two types of surface. An **open surface** has at least one **boundary curve** marking the furthest extent of the surface; an example is shown in Figure 30. A **closed surface** has no boundary curves, and divides three-dimensional space into two parts: the space inside the surface and the space outside the surface. (The shell of an egg forms a closed surface until you break it open.) In the case of an open surface, there are two choices for the set of unit normals, and it is necessary to state which choice has been made by, for example, drawing a diagram showing the unit normals at a few points. For any closed surface, a standard convention is used.

Unit normal convention for closed surfaces

For any closed surface, all the unit normals are chosen to point *outwards* into the exterior space, rather than inwards towards the enclosed volume (see Figure 31).

This convention has an important consequence for any vector field that represents a flow. If the flux of the field over a closed surface is positive, the net flow is outwards; if the flux is negative, the net flow is inwards.

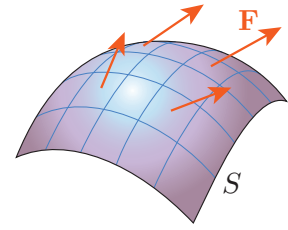


Figure 30 A vector field \mathbf{F} and an extended surface S

Note that the notation $d\mathbf{S}$ is used in surface integrals, while ds appears in line integrals.

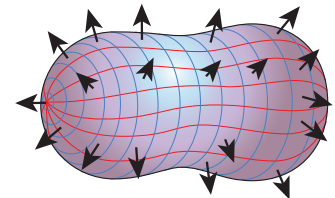


Figure 31 A closed surface and its outward-pointing unit normals

We now turn to the calculation of surface integrals of vector fields. Fortunately, the key ideas were introduced in Section 5 of Unit 8 in the context of finding the area of a curved surface.

The first idea is to label points on the surface by a pair of parameters, (u, v) . For example, points on the surface of a sphere, centred on the origin, can be labelled by the angular coordinates (θ, ϕ) of a spherical coordinate system. The radial coordinate r takes a constant value R all over the surface of the sphere. Since r does not distinguish different points on the surface, it is not regarded as one of the parameters that label points on the surface.

Points on the surface can also be labelled by their Cartesian coordinates (x, y, z) , and there is a set of equations linking (x, y, z) to (u, v) for any point on the surface:

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v).$$

On a spherical surface of radius R , for example, the relevant equations are

$$x = R \sin \theta \cos \phi, \quad y = R \sin \theta \sin \phi, \quad z = R \cos \theta.$$

This is reminiscent of the parametric description of a curve, but while each point on a curve is labelled by a single parameter t , each point on a curved surface is labelled by two parameters, (u, v) .

When u and v increase by tiny amounts δu and δv , a tiny patch is generated on the surface, as shown in Figure 32. Unit 8 showed that the area of such a patch is

$$\delta S = |\mathbf{J}| \delta u \delta v, \quad (34)$$

where \mathbf{J} is the *Jacobian vector*, given by

$$\mathbf{J} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}. \quad (35)$$

Unit 8 noted that this vector is perpendicular to the surface. We can always choose the order of the parameters u and v to ensure that \mathbf{J} is *parallel* to the chosen unit normal $\hat{\mathbf{n}}$ of the surface. Assuming that this has been done, the *oriented area* of the tiny surface patch is simply

$$\delta \mathbf{S} = \mathbf{J} \delta u \delta v,$$

and the flux of a vector field \mathbf{F} over the patch is given by

$$\mathbf{F} \cdot \delta \mathbf{S} = \mathbf{F} \cdot \mathbf{J} \delta u \delta v.$$

The flux over the entire surface S is found by adding up contributions like this from each of its patches. In the limit where the patches shrink to zero size, the sum becomes an integral over suitable ranges of u and v .

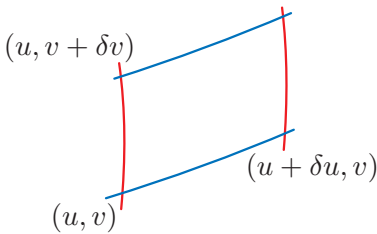


Figure 32 A patch generated by tiny increments in u and v

Evaluating the flux of a vector field over an extended surface

If points on a surface S are parametrised by (u, v) , where $u_1 \leq u \leq u_2$ and $v_1 \leq v \leq v_2$, then the flux of a vector field \mathbf{F} over the surface is given by the integral

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=v_1}^{v=v_2} \left(\int_{u=u_1}^{u=u_2} \mathbf{F} \cdot \mathbf{J} du \right) dv. \quad (36)$$

To evaluate this integral, the integrand $\mathbf{F} \cdot \mathbf{J}$ must be expressed in terms of the parameters u and v .

We assume that the limits of integration, u_1 , u_2 , v_1 and v_2 , are all constants.

To avoid duplicated effort, we skip the step of calculating \mathbf{J} from the determinant in equation (35). You have had practice at doing this in Unit 8, and there is nothing to be gained by going through similar steps again. To understand how equation (36) is used, it is sufficient to look at surfaces that are spheres or portions of spheres. The Jacobian vector \mathbf{J} for a spherical surface was calculated in Unit 8, and we quote that result here for ease of reference.

The calculation of \mathbf{J} for a spherical surface is in Exercise 26 of Unit 8.

The Jacobian vector \mathbf{J} on the surface of a sphere

On the surface of a sphere of radius R , centred on the origin and parametrised by (θ, ϕ) of spherical coordinates,

$$\mathbf{J} = R^2 \sin \theta \mathbf{e}_r, \quad (37)$$

where

$$\mathbf{e}_r = \sin \theta \cos \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \theta \mathbf{k} \quad (38)$$

is the radial unit vector of spherical coordinates.

This vector points radially outwards away from the centre of the sphere.

A simple but important case arises when a vector field expressed in spherical coordinates takes the form

$$\mathbf{F} = \frac{A}{r^2} \mathbf{e}_r \quad (r \neq 0),$$

where A is a constant. To calculate the flux of this field over a spherical surface of radius R , centred on the origin, we set $r = R$ in the expression for \mathbf{F} and take \mathbf{J} from equation (37). This gives

$$\mathbf{F} \cdot \mathbf{J} = \left(\frac{A}{R^2} \mathbf{e}_r \right) \cdot (R^2 \sin \theta \mathbf{e}_r) = A \sin \theta.$$

Since \mathbf{e}_r is a unit vector, $\mathbf{e}_r \cdot \mathbf{e}_r = 1$.

The flux of \mathbf{F} over the surface of the sphere is then given by

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= A \int_{\phi=0}^{\phi=2\pi} \left(\int_{\theta=0}^{\theta=\pi} \sin \theta d\theta \right) d\phi \\ &= A \int_{\phi=0}^{\phi=2\pi} [-\cos \theta]_{\theta=0}^{\theta=\pi} d\phi = 4\pi A. \end{aligned}$$

Remarkably enough, this flux does not depend on the radius of the sphere. This can be understood with very little calculation. The outward normal component of the field has the constant value A/R^2 all over the surface of the sphere. Hence all surface elements of the same area make the same contribution to the surface integral. Under these circumstances, the total flux of \mathbf{F} over the spherical surface can be found by multiplying the constant outward radial component of the field by the surface area of the sphere. This gives $A/R^2 \times 4\pi R^2 = 4\pi A$, in agreement with our more explicit calculation. Shortcuts like this are useful when the normal component of the field is constant at all points on the surface.

Example 10

Calculate the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ over the surface of a sphere of radius R , centred on the origin. You may use the standard integral

$$\int_0^\pi \sin^3 \theta \, d\theta = \frac{4}{3}.$$

Solution

The coordinate transformation equations for spherical coordinates are

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta.$$

On the surface of the sphere we have $r = R$, so on this surface the vector field is

$$\begin{aligned} \mathbf{F} &= R \sin \theta \cos \phi \mathbf{i} + R \sin \theta \sin \phi \mathbf{j} \\ &= R \sin \theta (\cos \phi \mathbf{i} + \sin \phi \mathbf{j}). \end{aligned}$$

The Jacobian vector on the surface of the sphere is

$$\mathbf{J} = R^2 \sin \theta \mathbf{e}_r,$$

so

$$\mathbf{F} \cdot \mathbf{J} = R^2 \sin^2 \theta (\cos \phi \mathbf{i} + \sin \phi \mathbf{j}) \cdot \mathbf{e}_r.$$

Using equation (38), we get

$$\mathbf{F} \cdot \mathbf{J} = R^3 \sin^3 \theta (\cos^2 \phi + \sin^2 \phi) = R^3 \sin^3 \theta.$$

To find the flux of \mathbf{F} over the surface of the sphere, we integrate $\mathbf{F} \cdot \mathbf{J}$ over the ranges $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$ that cover the sphere:

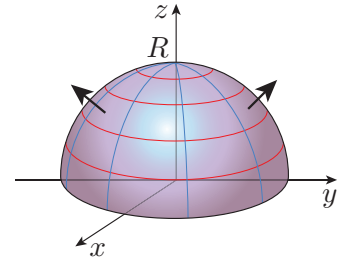
$$\int_S \mathbf{F} \cdot d\mathbf{S} = R^3 \int_{\phi=0}^{\phi=2\pi} \left(\int_{\theta=0}^{\theta=\pi} \sin^3 \theta \, d\theta \right) d\phi.$$

Using the standard integral given in the question, we conclude that

$$\int_S \mathbf{F} \cdot d\mathbf{S} = R^3 \int_{\phi=0}^{\phi=2\pi} \frac{4}{3} \, d\phi = \frac{8}{3} \pi R^3.$$

Exercise 14

Calculate the flux of the vector field $\mathbf{F} = z\mathbf{k}$ over the curved surface of a hemisphere of radius R shown in the margin, with its unit normals in the sense marked. (The flat base of the hemisphere is not included.)

**Exercise 15**

Calculate the flux of the vector field $\mathbf{F} = 3\mathbf{k}$ over the same hemispherical surface as in Exercise 14.

4.3 Divergence revisited

Unit 9 introduced the important concept of the divergence of a vector field:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \quad (39)$$

We claimed that this gives a measure of the extent to which \mathbf{F} diverges or flows away from any point. Various examples were used to illustrate this claim, but no proof was given. The concept of *flux* allows us to quantify this idea. If we surround a given point P by a tiny closed surface, the flux of a vector field \mathbf{F} over that surface gives us a measure of the flow of the field away from P . According to Unit 9, there should be a link between this flux and the divergence of the field at P . We now investigate this link.

Some discussion is needed to reach the main result – the *divergence theorem*. You should follow this discussion in outline to ensure that you understand the main ideas, but you will not be asked to reproduce the steps. The most important point is the divergence theorem itself (equation (43)) and its applications (e.g. Examples 11 and 12).

First, we choose a surface over which to calculate the flux. This choice will not affect our conclusions, but the working is simplified by using the surface of a tiny cube with sides of length δL , whose faces are aligned with the x -, y - and z -axes (see Figure 33). To find the flux over the surface of this cube, we must calculate the fluxes over each of its six faces and add them together. We take the faces in pairs, starting with the two shaded faces in Figure 33, which are perpendicular to the x -axis.

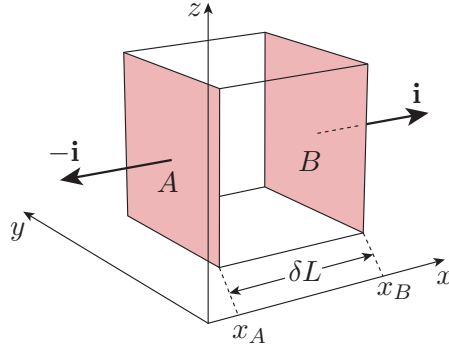


Figure 33 An enlarged view of a tiny cube with sides of length δL aligned with the x -, y - and z -axes

The faces are labelled A and B ; the left-hand face is at $x = x_A$, and the right-hand face is at $x = x_B$. It is important to recall that the unit normals of a closed surface always point *outwards* into the exterior space. This means that the unit normal for face A points in the negative x -direction and is equal to $-\mathbf{i}$, while the unit normal for face B is $+\mathbf{i}$. Consequently, the flux of \mathbf{F} over face A is

$$\text{flux over } A = \int_A \mathbf{F} \cdot d\mathbf{S} = \int_A (-F_x(x_A, y, z)) dy dz,$$

and the flux over face B is

$$\text{flux over } B = \int_B \mathbf{F} \cdot d\mathbf{S} = \int_B (+F_x(x_B, y, z)) dy dz.$$

Because the cube is aligned with the coordinate axes, the ranges of integration over y and z are identical in both these integrals. This allows us to express the total flux over both faces as an integral over the x - and y -values associated with face A :

$$\text{flux over } (A + B) = \int_A (F_x(x_B, y, z) - F_x(x_A, y, z)) dy dz.$$

Now, the integrand can be simplified. Dividing and multiplying by $x_B - x_A = \delta L$ and assuming that the cube is very small, we get

$$\begin{aligned} F_x(x_B, y, z) - F_x(x_A, y, z) &= \frac{F_x(x_B, y, z) - F_x(x_A, y, z)}{x_B - x_A} (x_B - x_A) \\ &\simeq \frac{\partial F_x}{\partial x} \delta L. \end{aligned}$$

We therefore obtain

$$\text{flux over } (A + B) \simeq \int_A \left(\frac{\partial F_x}{\partial x} \delta L \right) dy dz.$$

Because the cube is assumed to be very small, the integrand can be taken to be constant over face A . The integral is then just the product of the integrand and the area $(\delta L)^2$ of the face. We therefore conclude that

$$\text{flux over } (A + B) \simeq \frac{\partial F_x}{\partial x} (\delta L)^3 = \frac{\partial F_x}{\partial x} \delta V, \quad (40)$$

where δV is the volume of the cube.

There is nothing special about the x -axis. If we consider the pair of faces perpendicular to the y -axis, we get a similar result with x replaced everywhere by y . And if we consider the pair of faces perpendicular to the z -axis, we again get a similar result with x replaced everywhere by z . The total flux of \mathbf{F} over the entire surface of a small cube is found by adding these three contributions together:

$$\text{flux over surface of cube} \simeq \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \delta V.$$

Using the definition of divergence in Cartesian coordinates in equation (39), we see that

$$\text{flux over surface of cube} \simeq \nabla \cdot \mathbf{F} \delta V. \quad (41)$$

This is a remarkable result. It establishes the link between divergence and flux. All the approximations made in deriving it become exact in the limit as the volume of the cube shrinks to zero. The result has been derived in a special case, but it is true for all tiny elements of volume no matter what their shape. This allows us to think about divergence in a new way.

Divergence as flux per unit volume

The divergence of a vector field \mathbf{F} at a given point is related to the flux of \mathbf{F} over a tiny surface enclosing the point. In the limit where the surface area and its enclosed volume shrink to zero, we have

$$\nabla \cdot \mathbf{F} = \frac{\text{flux of } \mathbf{F} \text{ over surface}}{\text{volume enclosed by surface}}. \quad (42)$$

So the divergence of a vector field at any point can be interpreted as the *flux per unit volume* at that point.

4.4 Additivity of flux and the divergence theorem

The interpretation of divergence in equation (42) involves the limit of a tiny surface surrounding a point. With a little more effort, we can get a more powerful result – *the divergence theorem* – that applies over extended surfaces and is very useful in applications. To achieve this, we need to establish a rule that allows fluxes to be added together.

The additivity of flux

Suppose that a given volume is subdivided into smaller volume elements. Then the *additivity of flux* relates the flux of a vector field over the surface of the whole volume to its fluxes over the surfaces of the volume elements.

The additivity of flux

If a volume is subdivided into smaller volume elements, the flux of a vector field over the surface of the whole volume is the sum of its fluxes over the surfaces of all the volume elements.

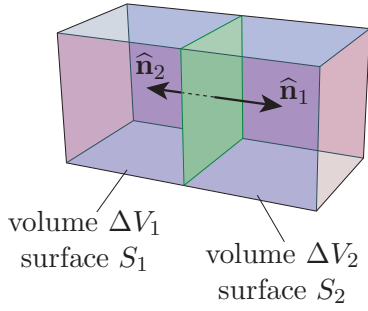


Figure 34 Neighbouring volume elements

To establish this fact, consider two neighbouring volume elements ΔV_1 and ΔV_2 with surfaces S_1 and S_2 (Figure 34). The surfaces S_1 and S_2 share a common boundary wall (shown in green). At any point on this boundary wall, the unit normal of S_1 points in the *opposite* direction to the unit normal of S_2 . This is because the unit normals of a closed surface always point outwards, away from the enclosed volume. It follows that the flux of a vector field \mathbf{F} contributed by the boundary wall section of S_1 is equal in magnitude and opposite in sign to the flux contributed by the boundary wall section of S_2 . So when we add up the fluxes of \mathbf{F} over the surfaces of all the volume elements, the contributions from the boundary walls all cancel out. The only surviving contributions come from the external surfaces, which form the surface of the whole volume.

The divergence theorem

Finally, we combine the additivity of flux with the interpretation of divergence as flux per unit volume. Suppose that we want to find the flux of a vector field \mathbf{F} over the surface S of a region V (which need not be small). Then we can divide the region into tiny subregions, with surfaces S_i . The additivity of flux tells us that

$$\int_S \mathbf{F} \cdot d\mathbf{S} \simeq \sum_i (\text{flux over } S_i),$$

where the sum is over the surfaces of all the volume elements that make up the region.

The volume elements are assumed to be tiny, so we can use equation (41) to express each flux in terms of divergence. This gives

$$\int_S \mathbf{F} \cdot d\mathbf{S} \simeq \sum_i \nabla \cdot \mathbf{F}(\mathbf{r}_i) \delta V_i.$$

Taking the limit of an infinite number of infinitesimal volume elements, the approximations become exact, and we conclude that

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F} dV.$$

This is the celebrated *divergence theorem*.

This is the key result of this section. It links surface integrals to related volume integrals.

Divergence theorem

Given a vector field \mathbf{F} and a closed surface S enclosing a volume V , the divergence theorem states that

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F} dV. \quad (43)$$

In other words, the surface integral of \mathbf{F} over a closed surface is equal to the volume integral of $\nabla \cdot \mathbf{F}$ over the interior of the surface.

It is easy to remember where the symbol ∇ goes. Divergence involves spatial derivatives, so its units are those of the field divided by length. To get the same units on both sides of equation (43), the divergence must be in the volume integral, rather than in the surface integral.

Origins of the divergence theorem

The divergence theorem is frequently called **Gauss's theorem**. In fact, it was discovered independently by several people: Joseph-Louis Lagrange in 1764, Carl Friedrich Gauss in 1813, George Green in 1828 and Mikhail Ostrogradsky in 1831.

The first two did not publish the theorem, but kept it in their personal papers. Green was an amateur mathematician with no connections to the academic world, and he published his findings in an obscure pamphlet. It was not until the 1830s that the theorem became well known, thanks to its applications in the newly-developing sciences of fluid mechanics and electromagnetism.

The divergence theorem allows us to convert tricky surface integrals into easier volume integrals. The following example illustrates this application.

Example 11

Use the divergence theorem to calculate the surface integral of $\mathbf{F} = 12z\mathbf{k}$ over the surface S of the rugby ball in Figure 35. The volume of this rugby ball was found to be $4\pi a^2b/3$ in Exercise 19 of Unit 8.

Solution

The divergence of \mathbf{F} is

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial(0)}{\partial x} + \frac{\partial(0)}{\partial y} + \frac{\partial(12z)}{\partial z} \\ &= 12.\end{aligned}$$

The divergence theorem allows us to express the required surface integral as a volume integral of $\nabla \cdot \mathbf{F}$ over the volume V of the rugby ball. Using the result given in the question, we get

$$\begin{aligned}\int_S \mathbf{F} \cdot d\mathbf{S} &= \int_V \nabla \cdot \mathbf{F} dV \\ &= \int_V 12 dV \\ &= 12 \times \frac{4}{3}\pi a^2b = 16\pi a^2b.\end{aligned}$$

With a slight modification of this technique, we can convert difficult surface integrals into easier ones.

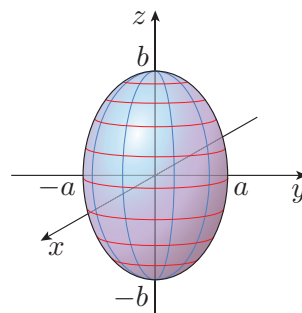


Figure 35 The surface of a rugby ball

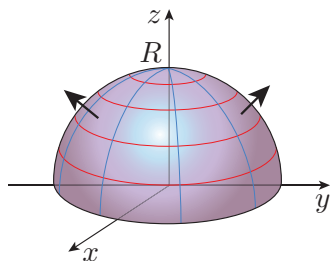


Figure 36 A hemispherical surface with its circular base in the xy -plane, centred on the origin

Example 12

Use the divergence theorem to find the flux of $\mathbf{F} = 3\mathbf{k}$ over the curved dome of the hemisphere in Figure 36, with unit normals as shown.

Solution

In order to apply the divergence theorem, we need a closed surface S . We take this to be the whole surface of the hemisphere, including the curved dome S_1 and the flat base S_2 , so

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_{S_1} \mathbf{F} \cdot d\mathbf{S} + \int_{S_2} \mathbf{F} \cdot d\mathbf{S},$$

where the normals of S_1 and S_2 both point outwards.

The field \mathbf{F} is constant, so its divergence is equal to zero everywhere. The divergence theorem then tells us that

$$\int_S \mathbf{F} \cdot d\mathbf{S} = 0,$$

so

$$\int_{S_1} \mathbf{F} \cdot d\mathbf{S} + \int_{S_2} \mathbf{F} \cdot d\mathbf{S} = 0.$$

Now, the surface integral over the flat base S_2 is trivial. Because the field \mathbf{F} is constant, and is perpendicular to this surface, we have

$$\int_{S_2} \mathbf{F} \cdot d\mathbf{S} = -3 \times (\pi R^2) = -3\pi R^2.$$

Here, the minus sign arises because the unit normals on the flat base point in the opposite direction to the field. Hence the required surface integral is

$$\int_{S_1} \mathbf{F} \cdot d\mathbf{S} = 3\pi R^2,$$

which agrees with the answer to Exercise 15.

Exercise 16

Use the divergence theorem to calculate the surface integral of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the surface S of a sphere of radius R .

Exercise 17

Use the divergence theorem to calculate the surface integral of the vector field $\mathbf{F} = z\mathbf{k}$ over the curved part of the hemispherical surface in Figure 36.

4.5 The equation of continuity

This subsection illustrates an important application of the divergence theorem, but will not be assessed. You can skip it if you are short of time.

Many quantities are neither created nor destroyed, but just move around from place to place. Such quantities are said to be **conserved**. For example, to a good approximation there is a fixed amount of air in the atmosphere. The air can move, and its local density may vary, but the total mass of air remains constant. We say that the mass of air is conserved. Something similar can be said about electric charge and energy. The divergence theorem allows us to express such behaviour in a precise way.

To take a definite case, we consider a fluid of constant total mass. At each point \mathbf{r} and time t , the fluid is described by its velocity $\mathbf{v}(\mathbf{r}, t)$ and its density $\rho(\mathbf{r}, t)$. We consider a fixed surface S enclosing a volume V . The total mass of fluid inside this surface is

$$M = \int_V \rho dV.$$

As the fluid moves around, the mass of fluid contained in the region V may change, and the rate of change of the enclosed mass is

$$\frac{dM}{dt} = \frac{d}{dt} \int_V \rho dV. \quad (44)$$

Naturally, we assume that there is no spontaneous creation or annihilation of fluid, so any change of fluid mass in the region V must be caused by a flow across the surface S . A net inward flow produces an increase in local mass, while a net outward flow leads to a decrease.

Equation (31) tells us that the rate of flow of fluid volume across a tiny planar element is $\mathbf{v} \cdot \delta\mathbf{S}$, where $\delta\mathbf{S}$ is the oriented area of the element. The corresponding rate of flow of fluid mass is $(\rho\mathbf{v}) \cdot \delta\mathbf{S}$. Hence the rate of flow of fluid mass out of the region V is given by the surface integral

$$\text{rate of outflow} = \int_S (\rho\mathbf{v}) \cdot d\mathbf{S}.$$

Because the unit normals of a closed surface point outwards, this is the rate of flow of fluid mass *out* of the region V , and is equal to $-dM/dt$, the rate of loss of mass from the enclosed region V . We therefore have

$$-\frac{dM}{dt} = \int_S (\rho\mathbf{v}) \cdot d\mathbf{S}. \quad (45)$$

Comparing equations (44) and (45), we conclude that

$$-\frac{d}{dt} \int_V \rho dV = \int_S (\rho\mathbf{v}) \cdot d\mathbf{S}. \quad (46)$$

This equation expresses the fact that any change in fluid mass in a region is related to a flow into or out of that region. The focus of interest here is that it can be recast in an alternative form using the divergence theorem.

First, the differentiation on the left-hand side can be brought inside the integral. This is allowed because the region of integration V does not change with time, so any change in the integral must be due to a change in its integrand. For any given region, the integral depends only on t , which is why straight dees have been used in equation (46). However, the

density ρ can depend on both t and \mathbf{r} , so we must use the curly dees of partial differentiation inside the integral. Thus

$$\frac{d}{dt} \int_V \rho dV = \int_V \frac{\partial \rho}{\partial t} dV.$$

We can also use the divergence theorem to express the right-hand side of equation (46) as a volume integral:

$$\int_S (\rho \mathbf{v}) \cdot d\mathbf{S} = \int_V \nabla \cdot (\rho \mathbf{v}) dV.$$

Equation (46) can therefore be written as

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) dV = 0, \quad (47)$$

where the two volume integrals over V have been combined into a single integral.

When a definite integral is equal to zero, it is normally unsafe to argue that its integrand must be equal to zero – there could, after all, be cancellations of positive and negative contributions. However, we are in a different position. Because equation (47) is valid for *any* region of integration, no matter how small or where it is located, the only possibility is for the integrand to be equal to zero everywhere. This leads to the following conclusion.

Equation of continuity

If the mass of a fluid is conserved, then at each point, its density ρ and velocity \mathbf{v} are related by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \quad (48)$$

This is known as the **equation of continuity** for fluid mass.

Equations like this appear throughout physics – wherever a quantity that flows like a fluid is conserved. For example, the flow of electric charge and the flow of energy both obey equations of continuity. You will meet this equation again in Unit 12 when we discuss diffusion.

We are sometimes interested in steady-state situations where the density ρ does not change in time at any point in the fluid. In this case, the equation of continuity gives $\nabla \cdot (\rho \mathbf{v}) = 0$. This restricts the possible flows that can occur in steady-state situations.

Exercise 18

Which of the following vector fields could describe $\rho \mathbf{v}$ in a steady-state flow in a fluid?

- (a) $\rho \mathbf{v} = 2y^2 \mathbf{i} - 14yz \mathbf{j} + 7z^2 \mathbf{k}$ (b) $\rho \mathbf{v} = 2x \mathbf{i} - 3y \mathbf{j} + 4z \mathbf{k}$

The vector field $\rho \mathbf{v}$ is sometimes called the *mass flux density*, and given the symbol \mathbf{J} .

5 Circulation and the curl theorem

This section brings our discussion of the calculus of fields to a close by investigating the meaning of the curl of a vector field. Its main result is the *curl theorem*, which is the counterpart of the divergence theorem of the previous section.

Recall that Unit 9 introduced the curl of a vector field as

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}. \quad (49)$$

This is a vector field. We claimed that the curl vector measures the extent to which \mathbf{F} is associated with rotation or swirling. A few examples supported this claim, but no proof was given. In this section, we use line integrals around closed paths to quantify the concept of curl.

5.1 Circulation of a vector field

We begin by establishing a convention. Figure 37 shows a planar surface element, with unit normal $\hat{\mathbf{n}}$. The perimeter of this element is a closed loop C , which we would like to treat as a path with some sense of positive progression. If the surface element were in the xy -plane, viewed from above, we might talk about progression in a clockwise or anticlockwise sense, but terms like this become ambiguous for planar elements with arbitrary orientations, viewed from arbitrary directions.

There are two possible choices for the unit normal of a planar element. A particular choice has been made in Figure 37. Having made this choice, we now fix the sense of positive progression around the perimeter curve C by the following convention.

Right-hand grip rule

See Figure 38: with the thumb of your right hand pointing in the direction of the unit normal of a planar element, the curled fingers of your right hand indicate the sense of positive progression around the perimeter of the element.

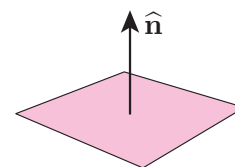


Figure 37 A planar surface element

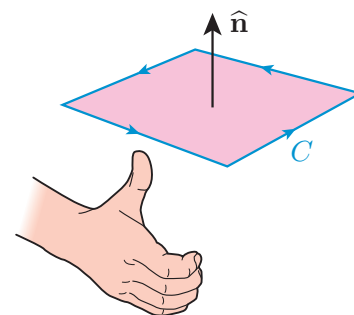
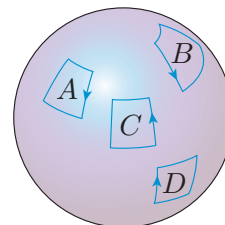


Figure 38 The right-hand grip rule

Exercise 19

The arrows in the figure in the margin show senses of progression around the perimeters of four shaded patches A , B , C and D on the surface of a sphere. For which of these patches do the arrows indicate a sense of positive progression?



Given a planar element $\delta\mathbf{S}$, we can define a closed path C that goes once round the perimeter of the element in the positive sense defined by the right-hand grip rule. Around this path, we can evaluate the line integral of a vector field \mathbf{F} . This line integral is called the *circulation* of the vector field around C . We write

$$\text{circulation} = \int_C \mathbf{F} \cdot d\mathbf{s} = \oint_C \mathbf{F} \cdot d\mathbf{s}.$$

The last expression has a circle in the middle of the integral sign. This symbol is sometimes used as a reminder that the path of integration is closed, but otherwise makes no difference to the meaning.

Circulation of a vector field

Given a vector field \mathbf{F} and a closed path C , the **circulation** of \mathbf{F} around C is given by the line integral

$$\text{circulation} = \oint_C \mathbf{F} \cdot d\mathbf{s}. \quad (50)$$

If C is a path around a planar element with a given unit normal, it is understood that C is traversed in the positive sense determined by the right-hand grip rule.

Section 2 explained how to calculate line integrals of this type. There is nothing new here except that the path is closed, which means that the start and end points of the path are identical. So if the path is described by the parametric equations

$$x = x(t), \quad y = y(t), \quad z = z(t) \quad (t_1 \leq t \leq t_2),$$

then the extreme parameter values t_1 and t_2 refer to the same point.

Exercise 20

- (a) Calculate the circulation of the vector field $\mathbf{F} = x\mathbf{j}$ around a closed path C with parametric representation

$$x(t) = \cos t, \quad y(t) = \sin t \quad (0 \leq t \leq 2\pi).$$

- (b) What is the circulation of a conservative vector field \mathbf{G} around the closed path C ?
-

5.2 Curl revisited

Circulation measures the amount of rotation or swirling associated with a vector field. You might therefore expect there to be a link between circulation and curl, similar to the link between flux and divergence. This is indeed the case, and we will now explore the nature of this link.

The following argument justifies the main result – the *curl theorem*. You should follow this discussion in outline to ensure that you understand the main ideas, but you will not be asked to reproduce the steps. The most important point is the curl theorem itself (equation (57)) and its applications (e.g. Examples 14 and 15).

We begin by considering a tiny square surface element with sides of length δL (Figure 39). The element lies in the xy -plane, with its edges aligned with the x - and y -axes, and its unit normal is chosen to be in the positive z -direction (towards you).

We will calculate the circulation of a vector field \mathbf{F} around the perimeter of this element. To do this, we must first use the right-hand grip rule to determine the positive sense of progression around the perimeter. This is in the order $ABCD$, as indicated by arrows in Figure 39. The path $ABCD$ consists of four straight-line segments, which we consider in pairs, starting with BC and DA , which vary in the y -direction.

Line integrals are usually evaluated in parametric form, but these paths are simple enough to make this unnecessary. We can revert to the fundamental concept of a line integral – that of integrating the component of a field in the direction of travel along a path. With the coordinates shown in Figure 39, the line integral contributed by the side BC is

$$I_{BC} = \int_{y_1}^{y_2} F_y(x_2, y, 0) dy,$$

and the contribution from the side DA is

$$I_{DA} = - \int_{y_1}^{y_2} F_y(x_1, y, 0) dy.$$

The minus sign on the right makes good sense because in the limit where x_1 approaches x_2 , the paths BC and DA become the reverse of one another, and must have opposite signs. The combined contribution from sides BC and DA is therefore

$$I_{BC+DA} = \int_{y_1}^{y_2} (F_y(x_2, y, 0) - F_y(x_1, y, 0)) dy.$$

Now, the integrand can be simplified. Dividing and multiplying by $x_2 - x_1 = \delta L$, and assuming that the square is very small, we get

$$\begin{aligned} F_y(x_2, y, 0) - F_y(x_1, y, 0) &= \frac{F_y(x_2, y, 0) - F_y(x_1, y, 0)}{x_2 - x_1} (x_2 - x_1) \\ &\simeq \frac{\partial F_y}{\partial x} \delta L, \end{aligned}$$

so

$$I_{BC+DA} \simeq \int_{y_1}^{y_2} \left(\frac{\partial F_y}{\partial x} \delta L \right) dy.$$

Because the square is assumed to be very small, the integrand can be taken to be constant over the range of integration. The integral is then approximated by the product of the integrand and the length $y_2 - y_1 = \delta L$.

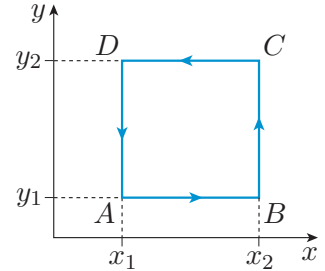


Figure 39 A square surface element in the xy -plane

So we have

$$I_{BC+DA} \simeq \frac{\partial F_y}{\partial x} (\delta L)^2. \quad (51)$$

A similar calculation can be done for the other two sides of the square. Their contribution is

$$I_{AB+CD} = \int_{x_1}^{x_2} (F_x(x, y_1, 0) - F_x(x, y_2, 0)) dx.$$

In this case, the x -component of the field appears in an integral over x , and the signs are different – the larger value of y (namely y_2) now appears in the term that carries a minus sign. Bearing these changes in mind, and working through the same steps as before, leads to

$$I_{AB+CD} \simeq \int_{x_1}^{x_2} \left(-\frac{\partial F_x}{\partial y} \delta L \right) dx = -\frac{\partial F_x}{\partial y} (\delta L)^2. \quad (52)$$

Finally, combining the contributions from equations (51) and (52), we conclude that the circulation of \mathbf{F} around the square $ABCD$ is

$$\text{circulation} \simeq \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \delta S, \quad (53)$$

where $\delta S = (\delta L)^2$ is the area of the square.

You may recognise the combination of partial derivatives in round brackets. Using the definition of curl in Cartesian coordinates in equation (49), you can see that

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = (\nabla \times \mathbf{F})_z = (\nabla \times \mathbf{F}) \cdot \mathbf{k}. \quad (54)$$

Because the square has its unit normal in the z -direction, its oriented area is $\delta \mathbf{S} = \delta S \mathbf{k}$. Combining this with equations (53) and (54), we see that

$$\text{circulation} \simeq (\nabla \times \mathbf{F}) \cdot \delta \mathbf{S}. \quad (55)$$

This is a remarkable result. It establishes the link we have been seeking between curl and circulation. All the approximations made in deriving it become exact in the limit where the area of the square shrinks to zero. The result has been derived in a particular case, but there is nothing special about our choice of axes, or the location of the square. In fact, equation (55) applies to all tiny planar elements of any shape. This enables us to think about curl in a new way.

Curl as circulation per unit area

Given a vector field \mathbf{F} in the vicinity of a given point, the component of $\nabla \times \mathbf{F}$ in the direction of the unit vector $\hat{\mathbf{n}}$ can be found by taking a planar element with unit normal $\hat{\mathbf{n}}$ at the point. The component is given by

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} = \frac{\text{circulation around perimeter of element}}{\text{area of element}}, \quad (56)$$

in the limit where the element becomes very small.

So each component of the curl at a given point can be interpreted as a *circulation per unit area* at that point.

Example 13

Consider the vector field

$$\mathbf{F} = -y\mathbf{i} + x\mathbf{j}.$$

- (a) Calculate the circulation of this vector field around the perimeter C of a tiny circular element of radius R , centred on the origin and lying in the xy -plane. The unit normal of this element is chosen to point in the positive z -direction.
- (b) Calculate the z -component of the curl of \mathbf{F} at the origin.
- (c) Do your answers to parts (a) and (b) agree with equation (56)?

Solution

- (a) Using the given direction of the unit normal, the right-hand grip rule tells us that the path must be traversed anticlockwise (when viewed from the positive z -axis). This path can be represented by parametric equations of the form

$$x = R \cos t, \quad y = R \sin t \quad (0 \leq t \leq 2\pi).$$

We have

$$\begin{aligned} \mathbf{F} &= -y\mathbf{i} + x\mathbf{j} = -R \sin t \mathbf{i} + R \cos t \mathbf{j}, \\ \frac{d\mathbf{s}}{dt} &= \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} = -R \sin t \mathbf{i} + R \cos t \mathbf{j}, \end{aligned}$$

so

$$\begin{aligned} \mathbf{F} \cdot \frac{d\mathbf{s}}{dt} &= F_x \frac{dx}{dt} + F_y \frac{dy}{dt} \\ &= R^2 (\sin^2 t + \cos^2 t) \\ &= R^2. \end{aligned}$$

The circulation around C is therefore

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} R^2 dt \\ &= 2\pi R^2. \end{aligned}$$

- (b) At any point, the z -component of the curl of \mathbf{F} is

$$\begin{aligned} (\nabla \times \mathbf{F}) \cdot \mathbf{k} &= \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \\ &= \frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y} \\ &= 2. \end{aligned}$$

In particular, $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = 2$ at the origin.

- (c) Because the area of the circular element is πR^2 , the right-hand side of equation (56) is equal to $2\pi R^2 / \pi R^2 = 2$. This is equal to the left-hand side, calculated at the centre of the element.

Exercise 21

Compare the two sides of equation (56) for a vector field $\mathbf{F} = x^2 \mathbf{j}$ and a square element in the xy -plane with corners A, B, C and D at (x, y) , $(x + a, y)$, $(x + a, y + a)$ and $(x, y + a)$, respectively, where a is a small constant length. The unit normal of the element is taken to be in the positive z -direction.

5.3 Additivity of circulation and the curl theorem

You have seen that the divergence of a vector field at a given point can be interpreted as the flux per unit volume in a tiny region around the point. Moreover, the additivity of flux allowed us to derive a more powerful result – the divergence theorem – which applies over *extended* regions. Now we will do something similar for curl.

The additivity of circulation

Consider an open surface in the plane of the page, divided into a number of subregions. As always, the unit normals of the subregions are required to have consistent orientations. To consider a definite case, we take them to point out of the page towards you. Then the right-hand grip rule ensures that the perimeters of the subregions are all traversed in the same sense – in this case, anticlockwise.

The *additivity of circulation* relates the circulation of a vector field around the surface to the sum of its circulations around the subregions.

The additivity of circulation

If an open surface is subdivided into consistently-oriented surface elements, the circulation of a vector field \mathbf{F} around the perimeter of the surface is the sum of the circulations of \mathbf{F} around the perimeters of all the elements.

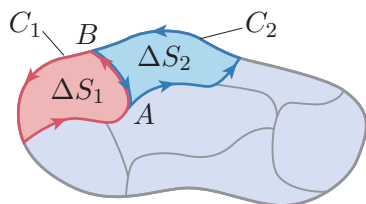


Figure 40 The additivity of circulation

To see why this is true, look at Figure 40; this shows two neighbouring subregions ΔS_1 and ΔS_2 , with perimeters C_1 (in red) and C_2 (in blue). These perimeters share a common segment AB , which is traversed in one sense for C_1 and in the *opposite* sense for C_2 . So when we add the circulations around C_1 and C_2 , the contributions from the common section AB cancel out. More generally, all the contributions from boundaries between subregions cancel out, leaving only contributions from sections that are not shared. But these non-shared sections form the perimeter of the whole surface.

In fact, the surface need not be flat. All we need is an open surface, such as that in Figure 41, divided into surface elements. (Recall that an open surface is one that has a perimeter.) If the surface elements are oriented consistently – with neighbouring elements having similar, rather than opposing, unit normals – the additivity of circulation continues to apply.

The curl theorem

Finally, we can combine the additivity of circulation with the interpretation of curl as circulation per unit area. Suppose that we want to find the circulation of a vector field \mathbf{F} around the perimeter C of an open surface. We can divide the surface into many tiny surface elements S_i with perimeters C_i . The additivity of circulation tells us that

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \sum_i (\text{circulation around } C_i),$$

where the sum is over the perimeters of all the surface elements that make up the surface.

The surface elements are assumed to be tiny, so we can use equation (55) to express each circulation in terms of curl. This gives

$$\oint_C \mathbf{F} \cdot d\mathbf{s} \simeq \sum_i (\nabla \times \mathbf{F}) \cdot \delta \mathbf{S}_i.$$

In the limit where the surface elements approach zero size, the approximation becomes exact and the right-hand side becomes an integral. We arrive at the following important result.

Curl theorem

If \mathbf{F} is a vector field and S is an open surface with perimeter C , then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}. \quad (57)$$

The curl theorem is just as important as the divergence theorem, and plays a central role in electromagnetism and fluid mechanics.

Origins of the curl theorem

The curl theorem is often called **Stokes's theorem** after the mathematician George Stokes (Figure 42), although the connection with Stokes is rather shaky.

The theorem was actually discovered by Lord Kelvin in 1850. Stokes learned about it in a letter from Kelvin, and set an exam question asking students to prove it. Exams must have been tough in those days! One of the students taking the exam was James Clerk Maxwell, who went on to use the theorem to help to frame the fundamental laws of electromagnetism. Stokes himself is famous for his work on fluid mechanics, wave motion and optics.

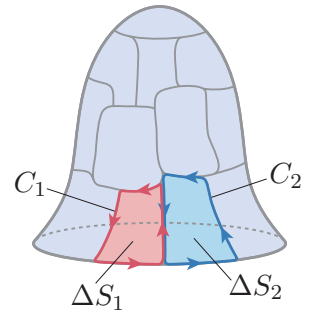


Figure 41 The additivity of circulation on a curved surface

This is the key result of this section. It links line integrals to related surface integrals.



Figure 42 George Stokes (1819–1903)

The curl theorem is useful because it can simplify the evaluation of integrals. For example, we can convert a line integral around a closed path into a simpler surface integral, as shown in the following example.

Example 14

Use the curl theorem to find the line integral of $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ around a circular path C in the xy -plane, centred on the origin and of radius R . The path is traversed anticlockwise when seen from the positive z -axis.

Solution

The curl of the two-dimensional vector field \mathbf{F} is

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{k} = \left(\frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y} \right) \mathbf{k} = 2\mathbf{k}.$$

The path C is the perimeter of a circular disc of radius R , centred on the origin and in the xy -plane. Since C is traversed in an anticlockwise sense, the right-hand grip rule shows that the unit normal of this surface is \mathbf{k} (rather than $-\mathbf{k}$).

Hence, using the curl theorem, the required line integral is

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{\text{disc}} 2\mathbf{k} \cdot \mathbf{k} dS = 2\pi R^2,$$

which agrees with the calculation in Example 13(a).

Exercise 22

Use the method of Example 14 to calculate the line integral of the vector field $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j}$ around a rectangular path in the xy -plane with corners at $(0,0)$, $(2,0)$, $(2,1)$, $(1,1)$, traversed in that order, and returning to $(0,0)$.

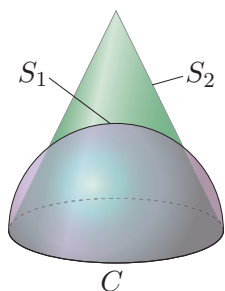


Figure 43 Two surfaces with the same perimeter path

It is worth noting that the curl theorem applies to all open surfaces, whether flat or not. In Figure 43, the surfaces S_1 and S_2 share the same perimeter path C . In this case, the curl theorem tells us that

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S},$$

for any vector field \mathbf{F} .

So if the vector field \mathbf{G} is the curl of \mathbf{F} (so that $\mathbf{G} = \nabla \times \mathbf{F}$), we have

$$\int_{S_1} \mathbf{G} \cdot d\mathbf{S} = \int_{S_2} \mathbf{G} \cdot d\mathbf{S}, \quad (58)$$

for any open surfaces S_1 and S_2 that share the same perimeter path.

This gives us the freedom to replace the surface integral of a vector field \mathbf{G} over a complicated surface by one over a much nicer surface – *but only if \mathbf{G} is a curl field* (i.e. a field that is the curl of another vector field). This is reminiscent of the freedom that we have to adjust the paths of *gradient fields* between fixed start and end points.

Example 15

The vector field $\mathbf{G} = z^2 \mathbf{j} + x^2 \mathbf{k}$ is a curl field. Use this fact to find the surface integral of \mathbf{G} over the curved hemispherical surface S in Figure 44, with its unit normals pointing upwards as shown.

Solution

The perimeter of the hemispherical surface is a circle in the xy -plane, centred on the origin and of radius R . The circular disc at the base of the hemisphere shares this perimeter path. The curl theorem can be applied to both these surfaces, but to ensure that the perimeter path is traversed in the same sense in both cases, the unit normal of the disc must be chosen to be \mathbf{k} (rather than $-\mathbf{k}$). Because \mathbf{G} is a curl field, we can replace the surface integral over the hemisphere by one over the disc.

Hence

$$\int_S \mathbf{G} \cdot d\mathbf{S} = \int_{\text{disc}} \mathbf{G} \cdot d\mathbf{S} = \int_{\text{disc}} (z^2 \mathbf{j} + x^2 \mathbf{k}) \cdot \mathbf{k} dS = \int_{\text{disc}} x^2 dS.$$

So we just need to integrate x^2 over a circular disc. Using polar coordinates, we get

$$\begin{aligned} \int_S \mathbf{G} \cdot d\mathbf{S} &= \int_{\phi=0}^{\phi=2\pi} \left(\int_{r=0}^{r=R} r^2 \cos^2 \phi \times r dr \right) d\phi \\ &= \int_0^{2\pi} \cos^2 \phi d\phi \times \int_0^R r^3 dr \\ &= \pi \times \frac{1}{4} R^4 = \frac{1}{4} \pi R^4. \end{aligned}$$

You can check that $\mathbf{G} = \nabla \times \mathbf{F}$, where $\mathbf{F} = \frac{1}{3}(z^3 \mathbf{i} + x^3 \mathbf{j})$.

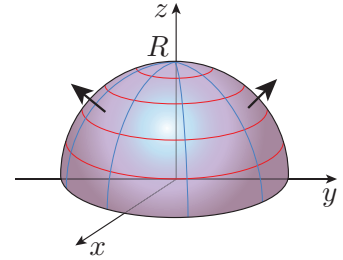
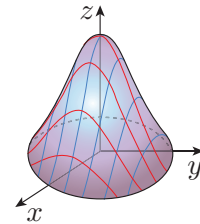


Figure 44 A hemispherical surface

Exercise 23

The vector field $\mathbf{G} = -2xz \mathbf{i} + (x^2 + z^2) \mathbf{k}$ is a curl field. Use the method of Example 15 to calculate the surface integral of \mathbf{G} over the curved surface of the bell shown in the margin. The open mouth of this bell is a circle in the xy -plane, centred on the origin and of radius R . The body of the bell lies in the region $z > 0$, and its unit normals point in the sense of increasing z .



Finally, there is some unfinished technical business. Subsection 3.3 proved that every conservative field has zero curl. The curl test assumes that the converse is true; so if the curl of \mathbf{F} is equal to zero throughout its domain, it assumes that \mathbf{F} is conservative. This is clearly unsafe logic: every owl is a bird, but every bird is not an owl! The curl test is usually reliable, but there is a proviso that can now be explained.

Suppose that \mathbf{F} is defined throughout the *whole of space*, and that $\nabla \times \mathbf{F} = \mathbf{0}$ *everywhere*. Then for any closed path C , the curl theorem tells us that

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0,$$

where S is an open surface with C as its perimeter.

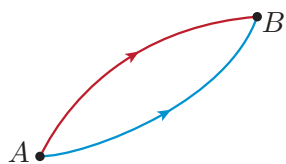


Figure 45 Two paths; reversing a path reverses the sign of a line integral

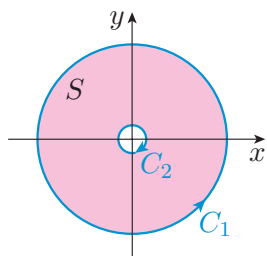


Figure 46 A case in which the domain of a two-dimensional vector field $\mathbf{F}(x, y)$ excludes the origin

Under these circumstances, we can say that \mathbf{F} has zero circulation around any closed loop C . It is easy to see that this implies that all the line integrals of \mathbf{F} are path-independent. For example, Figure 45 shows red and blue paths leading from A to B . The line integral of \mathbf{F} around a closed path C that follows the red path and the *reverse* of the blue path is

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{A \rightarrow B \text{ (red)}} \mathbf{F} \cdot d\mathbf{s} - \int_{A \rightarrow B \text{ (blue)}} \mathbf{F} \cdot d\mathbf{s}.$$

If this is equal to zero, the line integrals along the red and blue paths must be equal. This independence of path allows us to conclude that \mathbf{F} is conservative, and almost proves the curl test – but there is a loophole.

To see what can go wrong, consider the two-dimensional situation shown in Figure 46, where the vector field \mathbf{F} is not defined at the origin. If we consider a closed loop C_1 that encircles the origin, we can see that this is part of the boundary of an open surface S within the domain of \mathbf{F} . But it is not the complete boundary – there is also another portion C_2 , closer to the origin. Now the fact that $\nabla \times \mathbf{F} = \mathbf{0}$ tells us that the circulations around C_1 and C_2 cancel one another out – but they might not individually vanish. Under these circumstances, the curl test fails. In technical language, the curl test requires the domain of the field to be *simply-connected*.

A region is **simply-connected** if any closed loop in the region can be continuously shrunk to a point without leaving the region. For example, the whole of space is simply-connected. A typical Swiss cheese (with isolated holes) is also simply-connected. However, a plane with the origin removed, and three-dimensional space with the z -axis removed, are not simply-connected. In such domains, the condition $\nabla \times \mathbf{F} = \mathbf{0}$ does not guarantee the path-independence of all line integrals, so the curl test fails.

Learning outcomes

After studying this unit, you should be able to do the following.

- Calculate the line integral of a scalar field in Cartesian coordinates.
- Calculate the line integral of a vector field in Cartesian coordinates.
- State and apply the properties of conservative fields. Simplify line integrals involving conservative fields by choosing appropriate paths.
- Calculate the conservative vector field corresponding to a given scalar potential field, and find a scalar potential field corresponding to a given conservative vector field.
- Use the curl test to decide whether or not a given vector field is conservative.
- Define the terms closed surface, open surface, flux and oriented area, and understand the convention for the unit normals of a closed surface.

- Calculate the surface integral (or flux) of a vector field over a given surface (in simple cases).
- Interpret divergence as flux per unit volume. State and apply the additivity of flux and the divergence theorem.
- Use the right-hand grip rule to find the positive sense of progression around a given closed loop. Define and calculate circulation of a vector field.
- Interpret curl as flux per unit area. State and apply the additivity of circulation and the curl theorem.

Appendix: two insights

This Appendix is for interest and enjoyment only. It will not help you with calculations, but it contains two interesting insights that unify different topics in this book. You can read it when you have the time (possibly after studying the module).

Unifying various types of integral

In ordinary calculus, we can say that

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a), \quad (59)$$

a result known as the **fundamental theorem of calculus** because it brings together derivatives and integrals.

A similar result applies to gradients and line integrals:

$$\int_{A \rightarrow B} \nabla U \cdot d\mathbf{s} = U_B - U_A. \quad (60)$$

This is the content of equations (24) and (26), although the sign convention relating \mathbf{F} and U inserted minus signs in those equations. Equation (60) is sometimes called the **gradient theorem**.

Two other important results relating derivatives and integrals were discussed in this unit. The curl theorem can be written as

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{s}, \quad (61)$$

and the divergence theorem can be written as

$$\int_V \nabla \cdot \mathbf{F} dV = \int_S \mathbf{F} \cdot d\mathbf{S}. \quad (62)$$

There is a feature that unifies these four theorems. In each case, the left-hand side involves something that is differentiated and then integrated over a region, while the right-hand side contains no derivative, and is formed from values on the *boundary* of the region.

- In equation (59), the region is an interval along the x -axis, and its boundary is the pair of points $x = a$ and $x = b$ at either end of the interval.
- In equation (60), the region is a curved path, and its boundary consists of the pair of points A and B at either end of the path.
- In equation (61), the region is an open surface S , and its boundary is the closed path C that forms its perimeter.
- In equation (62), the region is a volume V , and its boundary is the surface S of this volume.

From this perspective, all of these theorems belong to the same family.

Expressions for divergence and curl in orthogonal coordinates

Unit 9 gave general formulas for divergence and curl in orthogonal coordinates. The optional Appendix of Unit 9 justified these formulas in a direct way, but it involved lengthy calculations. The divergence and curl theorems allow us to give alternative justifications that are simpler and more attractive.

According to Unit 9, in any orthogonal coordinate system (u, v, w) , divergence is given by

$$\nabla \cdot \mathbf{F} = \frac{1}{J} \left[\frac{\partial}{\partial u} \left(\frac{JF_u}{h_u} \right) + \frac{\partial}{\partial v} \left(\frac{JF_v}{h_v} \right) + \frac{\partial}{\partial w} \left(\frac{JF_w}{h_w} \right) \right], \quad (63)$$

where h_u , h_v and h_w are scale factors, and $J = h_u h_v h_w$ is the Jacobian factor.

The divergence theorem tells us that divergence can be interpreted as flux per unit volume. We can show that equation (63) is a direct expression of this fact. To see why, look at Figure 47, which shows a small volume element for the (u, v, w) orthogonal coordinate system. The volume of this element is

$$\delta V = h_u h_v h_w \delta u \delta v \delta w. \quad (64)$$

We need to calculate the flux of a vector field \mathbf{F} over the surface of the volume element. First, consider the two curved faces A (red) and B (green), which are perpendicular to the u -axis.

The calculation goes along the same lines as that given in Subsection 4.3 for Cartesian coordinates, but there is one significant difference. The faces A and B are generated by the same increments δv and δw , but they may have different areas. We must take account of this when we evaluate the fluxes. The area of each face is

$$\delta A = (h_v \delta v) \times (h_w \delta w), \quad (65)$$

and the faces have different areas if the scale factors h_u and h_w depend on u . Not surprisingly, equation (40) is replaced by

$$\text{flux over } (A + B) = \frac{\partial(F_u \delta A)}{\partial u} \delta u,$$

where the area of a face is now *inside* the partial derivative.

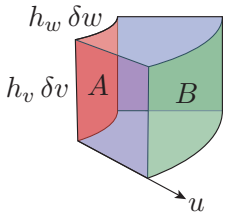


Figure 47 A volume element in (u, v, w) coordinates

Using equation (65) and recalling that δv and δw are the same for both faces, we get

$$\text{flux over } (A + B) = \frac{\partial(F_u h_v h_w)}{\partial u} \delta u \delta v \delta w = \frac{\partial}{\partial u} \left(\frac{J F_u}{h_u} \right) \delta u \delta v \delta w.$$

Of course, there are similar expressions for the other two pairs of faces, perpendicular to the v - and w -axes. Adding together all these fluxes gives the total flux over the surface of the volume element:

$$\text{total flux} = \left[\frac{\partial}{\partial u} \left(\frac{J F_u}{h_u} \right) + \frac{\partial}{\partial v} \left(\frac{J F_v}{h_v} \right) + \frac{\partial}{\partial w} \left(\frac{J F_w}{h_w} \right) \right] \delta u \delta v \delta w. \quad (66)$$

Divergence is flux per unit volume. We therefore divide equation (66) by equation (64), and take the limit of a tiny volume element. In this limit, all our approximations become exact, and we recover equation (63).

We can also justify the formula for curl in orthogonal coordinates. In any right-handed orthogonal coordinate system (u, v, w) , with scale factors h_u , h_v and h_w , Unit 9 gave the following formula for curl:

$$\nabla \times \mathbf{F} = \frac{1}{h_u h_v h_w} \begin{vmatrix} h_u \mathbf{e}_u & h_v \mathbf{e}_v & h_w \mathbf{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}. \quad (67)$$

Figure 48 shows a tiny area element based on the orthogonal coordinates (u, v, w) . This element is perpendicular to the w -axis, and we can obtain an expression for the w -component of $\nabla \times \mathbf{F}$ by finding the circulation per unit area of \mathbf{F} around it.

The calculation goes along similar lines to that given in Subsection 5.2 for Cartesian coordinates. The main new feature is that opposite sides of the element need not be equal in length. Their lengths depend on scale factors, which may vary from point to point. Taking this into account, it is not difficult to show that the circulation around the element in Figure 48 is

$$\text{circulation} = \left(\frac{\partial(h_v F_v)}{\partial u} - \frac{\partial(h_u F_u)}{\partial v} \right) \delta u \delta v, \quad (68)$$

while the area of the element is

$$\delta A = h_u h_v \delta u \delta v.$$

Curl is circulation per unit volume. We therefore divide equation (68) by δA , and take the limit of a tiny area element. In this limit, all our approximations become exact, and we recover the w -component of equation (67). Similar arguments give the other two components of the curl.

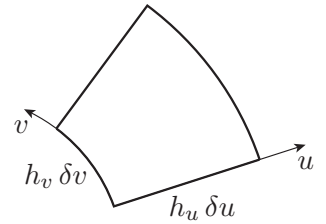


Figure 48 A surface element in the (u, v, w) coordinate system

Solutions to exercises

Solution to Exercise 1

We have $dx/dt = 2$ and $dy/dt = 2t$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2^2 + (2t)^2 = 4(1 + t^2).$$

The length of the parabolic arc is

$$L = \int_{-1}^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = 2 \int_{-1}^1 \sqrt{1 + t^2} dt.$$

Using the standard integral given in the question, we conclude that

$$\begin{aligned} L &= \left[t\sqrt{1+t^2} + \ln(t + \sqrt{1+t^2}) \right]_{t=-1}^{t=1} \\ &= 2\sqrt{2} + \ln(\sqrt{2} + 1) - \ln(\sqrt{2} - 1) \simeq 4.59. \end{aligned}$$

Solution to Exercise 2

We have

$$\frac{dx}{dt} = -a \sin t, \quad \frac{dy}{dt} = a \cos t, \quad \frac{dz}{dt} = b,$$

so

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 &= (-a \sin t)^2 + (a \cos t)^2 + b^2 \\ &= a^2 + b^2. \end{aligned}$$

The length of the helical path is therefore

$$L = \int_0^{2\pi} \sqrt{a^2 + b^2} dt = 2\pi\sqrt{a^2 + b^2}.$$

Check: In the limit where $b = 0$, our answer reduces to $2\pi a$, which is the circumference of a circle, as expected.

Solution to Exercise 3

The total number of ants is given by the line integral

$$N = \int_C \lambda dl.$$

From the given parametric representation, we have

$$\lambda(x(t), y(t)) = \frac{A}{R^4} (R \cos t)^2 (R \sin t) = \frac{A}{R} \cos^2 t \sin t.$$

Also,

$$\frac{dx}{dt} = -R \sin t, \quad \frac{dy}{dt} = R \cos t,$$

so

$$\delta l \simeq \sqrt{(-R \sin t)^2 + (R \cos t)^2} \delta t = R \delta t.$$

Hence

$$N = \int_0^\pi \frac{A}{R} \cos^2 t \sin t \times R dt = A \int_0^\pi \cos^2 t \sin t dt.$$

The integral can be completed by substituting $u = \cos t$. Then $du/dt = -\sin t$. The lower limit $t = 0$ corresponds to $u = \cos 0 = 1$, and the upper limit $t = \pi$ corresponds to $u = \cos \pi = -1$, so

$$\begin{aligned} N &= A \int_{t=0}^{t=\pi} u^2 \left(-\frac{du}{dt} \right) dt \\ &= -A \int_{u=1}^{u=-1} u^2 du = -A \left[\frac{1}{3} u^3 \right]_1^{-1} = \frac{2}{3} A. \end{aligned}$$

Solution to Exercise 4

We have $dr/dt = 2$ and $d\phi/dt = 1$, so

$$L = \int_0^5 \sqrt{4 + 4t^2 \times 1} dt = 2 \int_0^5 \sqrt{1 + t^2} dt.$$

Using the standard integral given in Exercise 1, we get

$$\begin{aligned} L &= \left[t\sqrt{1+t^2} + \ln(t + \sqrt{1+t^2}) \right]_0^5 \\ &= 5\sqrt{26} + \ln(\sqrt{26} + 5) \simeq 27.8. \end{aligned}$$

Solution to Exercise 5

Along path A , $d\theta/dt = 1$ and $d\phi/dt = 0$, so equation (14) gives

$$L = R \int_0^{\pi/2} \sqrt{1} dt = \frac{\pi}{2} R.$$

Along path B , $d\theta/dt = 0$, $d\phi/dt = 1$ and $\sin \theta = \sin(\pi/6) = \frac{1}{2}$, so

$$L = R \int_0^{\pi/2} \sqrt{\frac{1}{4}} dt = \frac{\pi}{4} R.$$

Solution to Exercise 6

Differentiating the parametric equations $x = 2 - t$ and $y = t$ gives

$$\frac{dx}{dt} = -1, \quad \frac{dy}{dt} = 1.$$

Expressing the components of \mathbf{F} in terms of t , we get

$$\begin{aligned} F_x &= x - y = (2 - t) - t = 2(1 - t), \\ F_y &= x + y = (2 - t) + t = 2. \end{aligned}$$

So

$$\mathbf{F} \cdot \frac{d\mathbf{s}}{dt} = F_x \frac{dx}{dt} + F_y \frac{dy}{dt} = -2(1 - t) + 2 = 2t.$$

The required line integral is

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{s} = \int_0^2 2t dt = [t^2]_0^2 = 4.$$

Solution to Exercise 7

Differentiating the parametric equations gives

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2, \quad \frac{dz}{dt} = 0.$$

Expressing the components of \mathbf{F} in terms of t , we get

$$F_x = yz = 4(1 + 2t), \quad F_y = xz = 4t, \quad F_z = xy = t(1 + 2t).$$

Hence

$$\mathbf{F} \cdot \frac{d\mathbf{s}}{dt} = 4(1 + 2t) + 8t = 4 + 16t.$$

The required line integral is therefore

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (4 + 16t) dt = [4t + 8t^2]_0^1 = 12.$$

Solution to Exercise 8

Differentiating the parametric equations gives

$$\frac{dx}{dt} = -1, \quad \frac{dy}{dt} = -2, \quad \frac{dz}{dt} = 0.$$

Expressing the components of \mathbf{F} in terms of t , we get

$$F_x = yz = 4(3 - 2t), \quad F_y = xz = 4(1 - t), \quad F_z = xy = (1 - t)(3 - 2t).$$

Hence

$$\mathbf{F} \cdot \frac{d\mathbf{s}}{dt} = -4(3 - 2t) - 8(1 - t) = -20 + 16t.$$

The required line integral is therefore

$$\int_{C_{\text{rev}}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (-20 + 16t) dt = [-20t + 8t^2]_0^1 = -12,$$

which is minus the answer of Exercise 7, as expected.

Solution to Exercise 9

Because \mathbf{F} is a gradient field, any line integral with start point $(1, 1)$ and end point $(7, 3)$ has value

$$\int_C \mathbf{F} \cdot d\mathbf{s} = U(1, 1) - U(7, 3) = 0 - (-20) = 20.$$

Solution to Exercise 10

Because the field is conservative, we can choose any convenient path with the given start and end points. A simple choice is the straight-line path C from the origin $(0, 0)$ to $(1, 2)$. Along this path $y = 2x$, so a suitable parametrisation is

$$x = t, \quad y = 2t \quad (0 \leq t \leq 1).$$

Then we have $dx/dt = 1$ and $dy/dt = 2$, so

$$\begin{aligned}\mathbf{F} \cdot \frac{d\mathbf{s}}{dt} &= F_x \frac{dx}{dt} + F_y \frac{dy}{dt} = (3t^2 \times 2t) \times 1 + (t^3 + 8t^3) \times 2 \\ &= 6t^3 + 18t^3 = 24t^3.\end{aligned}$$

So the line integral is

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 24t^3 dt = 6.$$

Solution to Exercise 11

The scalar potential field is given by

$$U(\mathbf{r}) = - \int_{\mathbf{0} \rightarrow \mathbf{r}} \mathbf{F} \cdot d\mathbf{s}.$$

We consider an arbitrary point $\mathbf{r} = (a, b)$, and choose a straight-line path from the origin to this point. This path can be described by the parametric equations

$$x = at, \quad y = bt \quad (0 \leq t \leq 1).$$

The values of a and b are constant along the path, so

$$\frac{dx}{dt} = a \quad \text{and} \quad \frac{dy}{dt} = b.$$

Hence

$$\mathbf{F} \cdot \frac{d\mathbf{s}}{dt} = F_x \frac{dx}{dt} + F_y \frac{dy}{dt} = a \cos(at) + b \sin(bt)$$

and

$$\begin{aligned}U(a, b) &= - \int_{t=0}^{t=1} (a \cos(at) + b \sin(bt)) dt \\ &= - [\sin(at) - \cos(bt)]_{t=0}^{t=1} \\ &= \cos b - \sin a - 1.\end{aligned}$$

However, the point (a, b) is arbitrary, so for any point (x, y) ,

$$U(x, y) = \cos y - \sin x - 1.$$

This answer can be checked by taking its gradient:

$$\nabla U = \frac{\partial U}{\partial x} \mathbf{i} + \frac{\partial U}{\partial y} \mathbf{j} = -\cos x \mathbf{i} - \sin y \mathbf{j} = -\mathbf{F}.$$

Solution to Exercise 12

(a) We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & z \end{vmatrix} = \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(1-1) = \mathbf{0}.$$

So the curl test shows that \mathbf{F} is conservative.

(b) We have

$$\nabla \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(1+1).$$

This is not equal to $\mathbf{0}$ everywhere, so \mathbf{G} is *not* conservative.

Solution to Exercise 13

For element A , the unit normal is \mathbf{i} , so the normal component of the field is $\mathbf{i} \cdot \mathbf{F} = 2$. This element has area 2, so the flux over it is equal to 4.

For element B , the unit normal is $-\mathbf{j}$, so the normal component of the field is $-\mathbf{j} \cdot \mathbf{F} = -3$. This element has area 1, so the flux over it is equal to -3 .

Solution to Exercise 14

The coordinate transformation equations for spherical coordinates are

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta.$$

So on the curved surface of the hemisphere, where $r = R$, the vector field is

$$\mathbf{F} = z \mathbf{k} = R \cos \theta \mathbf{k}.$$

The unit normals shown in the figure point in the same outward direction as those for a complete sphere, so the formula for \mathbf{J} in equation (37) can be used for the hemisphere. We therefore have

$$\begin{aligned} \mathbf{F} \cdot \mathbf{J} &= (R \cos \theta \mathbf{k}) \cdot (R^2 \sin \theta \mathbf{e}_r) \\ &= R^3 \cos \theta \sin \theta \mathbf{k} \cdot \mathbf{e}_r \\ &= R^3 \cos^2 \theta \sin \theta. \end{aligned}$$

From equation (38),
 $\mathbf{k} \cdot \mathbf{e}_r = \cos \theta$.

The surface of the hemisphere is defined by $0 \leq \theta \leq \pi/2$ and $0 \leq \phi \leq 2\pi$, so the required flux is

$$\int_S \mathbf{F} \cdot d\mathbf{S} = R^3 \int_{\phi=0}^{\phi=2\pi} \left(\int_{\theta=0}^{\theta=\pi/2} \cos^2 \theta \sin \theta d\theta \right) d\phi.$$

To carry out the integral over θ , we make the substitution $u = \cos \theta$. Then $du/d\theta = -\sin \theta$. The limit $\theta = 0$ corresponds to $u = 1$, and the limit $\theta = \pi/2$ corresponds to $u = 0$, so

$$\begin{aligned} \int_{\theta=0}^{\theta=\pi/2} \cos^2 \theta \sin \theta d\theta &= \int_{\theta=0}^{\theta=\pi/2} u^2 \left(-\frac{du}{d\theta} \right) d\theta \\ &= - \int_1^0 u^2 du = \frac{1}{3}. \end{aligned}$$

Hence

$$\int_S \mathbf{F} \cdot d\mathbf{S} = R^3 \int_0^{2\pi} \frac{1}{3} d\phi = \frac{2}{3}\pi R^3.$$

Solution to Exercise 15

Following the same method as in Exercise 14, we have

$$\mathbf{F} = 3\mathbf{k} \quad \text{and} \quad \mathbf{J} = R^2 \sin \theta \mathbf{e}_r,$$

so

$$\mathbf{F} \cdot \mathbf{J} = 3R^2 \sin \theta \mathbf{k} \cdot \mathbf{e}_r = 3R^2 \sin \theta \cos \theta = \frac{3}{2}R^2 \sin(2\theta).$$

The flux over the hemispherical surface is

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= \frac{3}{2}R^2 \int_{\phi=0}^{\phi=2\pi} \left(\int_{\theta=0}^{\theta=\pi/2} \sin(2\theta) d\theta \right) d\phi \\ &= \frac{3}{2}R^2 \int_{\phi=0}^{\phi=2\pi} \left[-\frac{1}{2} \cos(2\theta) \right]_0^{\pi/2} d\phi \\ &= \frac{3}{2}R^2 \int_0^{2\pi} 1 d\phi = 3\pi R^2. \end{aligned}$$

Solution to Exercise 16

The divergence of \mathbf{F} is

$$\nabla \cdot \mathbf{F} = \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z} = 3.$$

Using the divergence theorem, the required surface integral is

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F} dV = \int_V 3 dV,$$

where the volume integral is over the volume of a sphere of radius R .

Hence

$$\int_S \mathbf{F} \cdot d\mathbf{S} = 3 \times \frac{4}{3}\pi R^3 = 4\pi R^3.$$

Solution to Exercise 17

As in Example 12, we consider the whole surface of the hemisphere, S , which includes the curved dome S_1 and the flat base S_2 . Then

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_{S_1} \mathbf{F} \cdot d\mathbf{S} + \int_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

On the flat base of the hemisphere, $z = 0$, so in this case

$$\int_{S_2} \mathbf{F} \cdot d\mathbf{S} = 0$$

and

$$\int_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F} dV,$$

where V is the region bounded by the hemispherical surface and its base.

The divergence of $\mathbf{F} = z \mathbf{k}$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial(0)}{\partial x} + \frac{\partial(0)}{\partial y} + \frac{\partial(z)}{\partial z} = 1,$$

so

$$\int_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_V 1 \, dV = \frac{2}{3}\pi R^3,$$

where we have used the fact that a hemisphere of radius R has half the volume of a complete sphere (i.e. $\frac{1}{2} \times \frac{4}{3}\pi R^3$). The answer agrees with that of Exercise 14.

Solution to Exercise 18

- (a) In order for the flow to be steady-state, we must have $\nabla \cdot (\rho \mathbf{v}) = 0$. In this case,

$$\nabla \cdot (\rho \mathbf{v}) = \frac{\partial(2y^2)}{\partial x} + \frac{\partial(-14yz)}{\partial y} + \frac{\partial(7z^2)}{\partial z} = 0 - 14z + 14z = 0,$$

so this can be a steady-state flow.

- (b) In this case,

$$\nabla \cdot (\rho \mathbf{v}) = \frac{\partial(2x)}{\partial x} + \frac{\partial(-3y)}{\partial y} + \frac{\partial(4z)}{\partial z} = 2 - 3 + 4 = 3 \neq 0,$$

so this cannot be a steady-state flow.

Solution to Exercise 19

The unit normals of a closed surface all point outwards into the exterior space. Using the right-hand grip rule, we see that the perimeters of B and C are traversed in a positive sense, and the perimeters of A and D are traversed in a negative sense.

Solution to Exercise 20

- (a) We have

$$\frac{d\mathbf{s}}{dt} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} = -\sin t \mathbf{i} + \cos t \mathbf{j}$$

and

$$\mathbf{F} = \cos t \mathbf{j}.$$

Hence

$$\mathbf{F} \cdot \frac{d\mathbf{s}}{dt} = \cos^2 t,$$

and the circulation of \mathbf{F} around C is

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \cos^2 t \, dt = \int_0^{2\pi} \frac{1}{2}(1 + \cos(2t)) \, dt = \pi.$$

- (b) The line integral of a conservative field around a closed path is equal to zero, so the circulation of \mathbf{G} around C is equal to zero.

Solution to Exercise 21

The left-hand side of equation (56) is $(\nabla \times \mathbf{F}) \cdot \mathbf{k}$, which is equal to

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial(x^2)}{\partial x} - \frac{\partial(0)}{\partial y} = 2x.$$

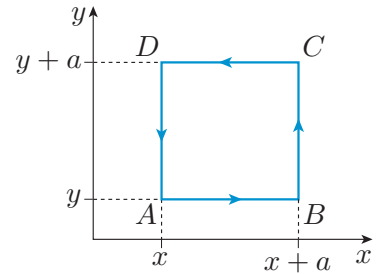
With the unit normal in the positive z -direction, the perimeter of $ABCD$ must be traversed in an anticlockwise sense, as shown in the diagram. The field acts in the y -direction, so only sides BC and DA contribute to the circulation. Along side BC , the component of the field in the direction of the path has the constant value $(x+a)^2$. When this is integrated along the length of BC , it makes a contribution $(x+a)^2a$ to the circulation. Along side DA , the component of the field in the direction of the path has the constant value $-x^2$. When this is integrated along the length of DA , it makes a contribution $-x^2a$ to the circulation. So the total circulation of \mathbf{F} around $ABCD$ is

$$\text{circulation} = (x+a)^2a - x^2a = (x^2 + 2ax + a^2)a - x^2a = 2a^2x + a^3.$$

The area of the square is a^2 , so

$$\text{circulation per unit area} = \frac{2a^2x + a^3}{a^2} = 2x + a.$$

In the limit where a tends to zero, the left- and right-hand sides of equation (56) become equal, as required. (Note that in general, equation (56) applies only in the limit of a vanishingly small element.)



Solution to Exercise 22

Using the curl theorem, the required line integral can be calculated from the surface integral of $\nabla \times \mathbf{F}$ over a rectangular area. However, we have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & 0 \end{vmatrix} = \mathbf{0},$$

so the surface integral, and hence the required line integral, is equal to zero.

Solution to Exercise 23

Because \mathbf{F} is a curl field, we can replace the curved surface of the bell by the flat circular disc at its mouth. To ensure that both surfaces have the same perimeter path, traversed in the same sense, the unit normal of the disc must be taken to be \mathbf{k} (rather than $-\mathbf{k}$). We therefore get

$$\int_{\text{bell}} \mathbf{G} \cdot d\mathbf{S} = \int_{\text{disc}} \mathbf{G} \cdot d\mathbf{S} = \int_{\text{disc}} (-2xz\mathbf{i} + (x^2 + z^2)\mathbf{k}) \cdot \mathbf{k} dS.$$

The disc lies in the xy -plane, so $z = 0$. Hence the integral reduces to

$$\int_{\text{bell}} \mathbf{G} \cdot d\mathbf{S} = \int_{\text{disc}} x^2 dS.$$

This integral was evaluated in Example 15, so the answer is $\pi R^4/4$.

Although it is not part of the question, you could check that $\mathbf{G} = \nabla \times \mathbf{F}$, where $\mathbf{F} = -x^2y\mathbf{i} + xz^2\mathbf{j}$.

Acknowledgements

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